N-Gram Language Models

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UIC CS 421

Language is inherently contextual.

- Words or characters in language are dependent upon one another!
- **Sequence modeling** allows us to make use of sequential information in language
- What are some ways we can model sequences?
  - **Language models**
  - **Hidden Markov models**

This Week's Topics

Tuesday

- N-gram language modeling
- Evaluating LMs
- Improving n-gram LMs

Thursday

- Hidden Markov Models
- Forward Algorithm
- Viterbi Algorithm
- Forward-Backward Algorithm

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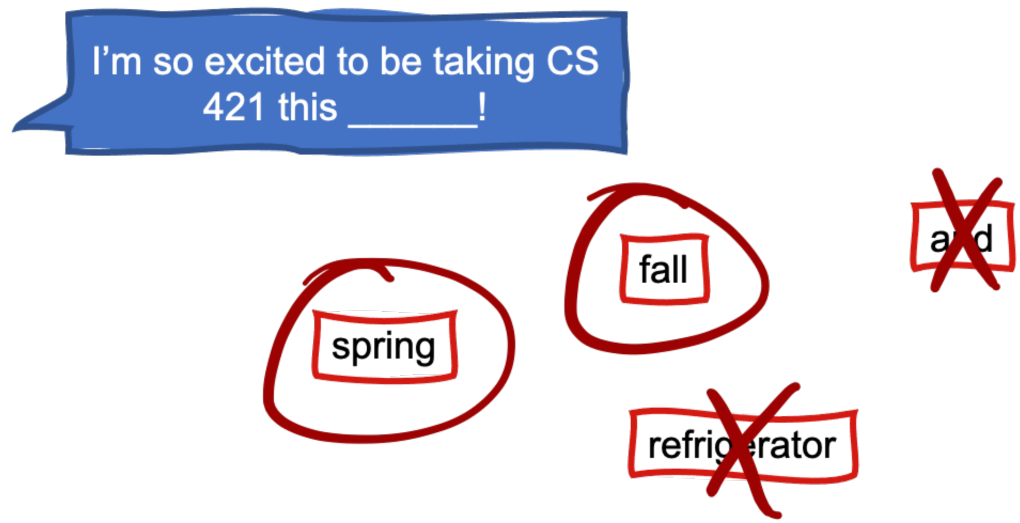
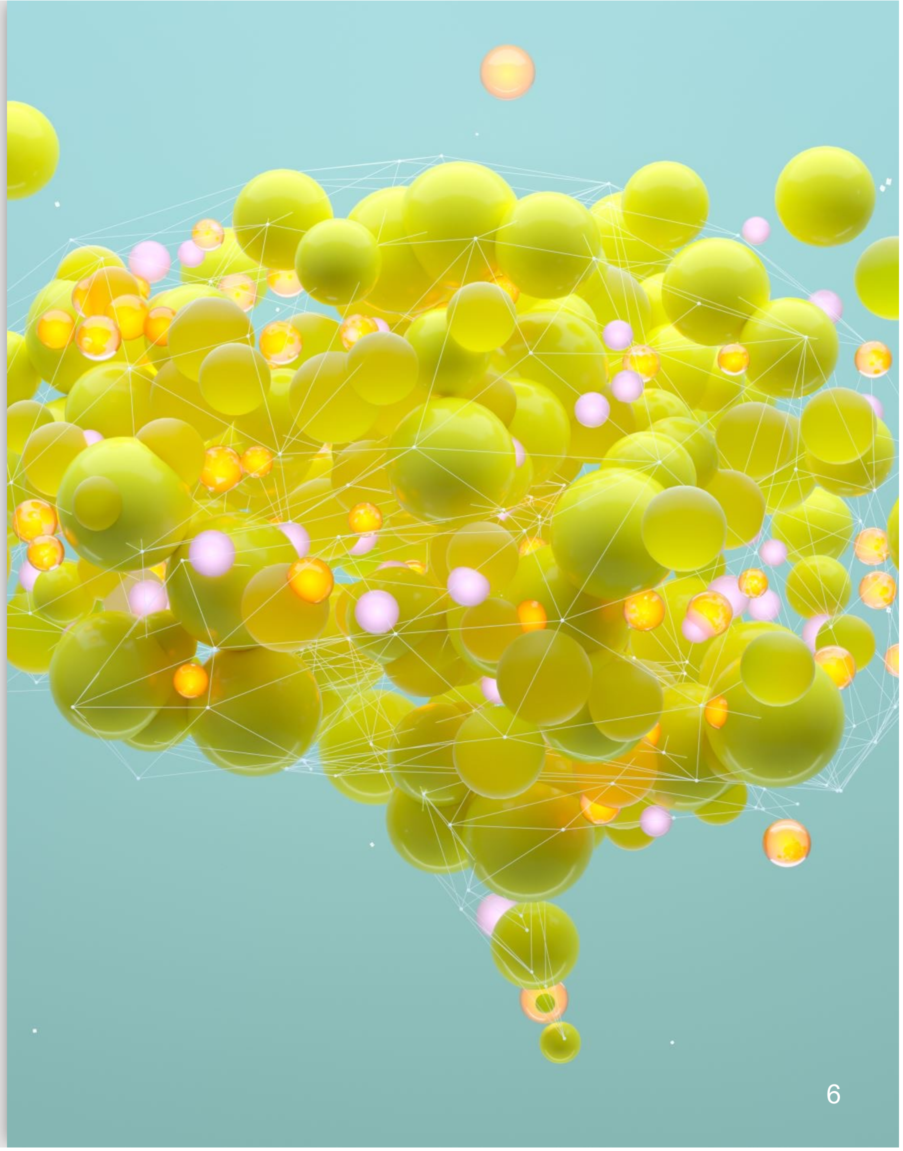
Language Modeling

- Learning how to effectively predict the likelihood of word or character sequences in a language



Why is language modeling useful?

- Helps identify words in noisy, ambiguous input
  - Speech recognition or autocorrect
- Helps generate natural-sounding language
  - Machine translation or image captioning
- In contemporary NLP, language modeling forms the basis of most approaches
  - Language representation



Language models come in many forms!

- Better with small datasets:
  - **N-gram language models**
- More sophisticated
  - Neural language models

N-Grams

- Sequences of a predefined item type within a language
  - N → Size of the sequence
  - -gram → Greek-derived suffix meaning "what is written"
- First use of the term appears to be in the late 1940s
  - *A Mathematical Theory of Communication*, by Claude Shannon: https://people.math.harvard.edu/~ctm/home/text/others/shannon/entropy/entropy.pdf

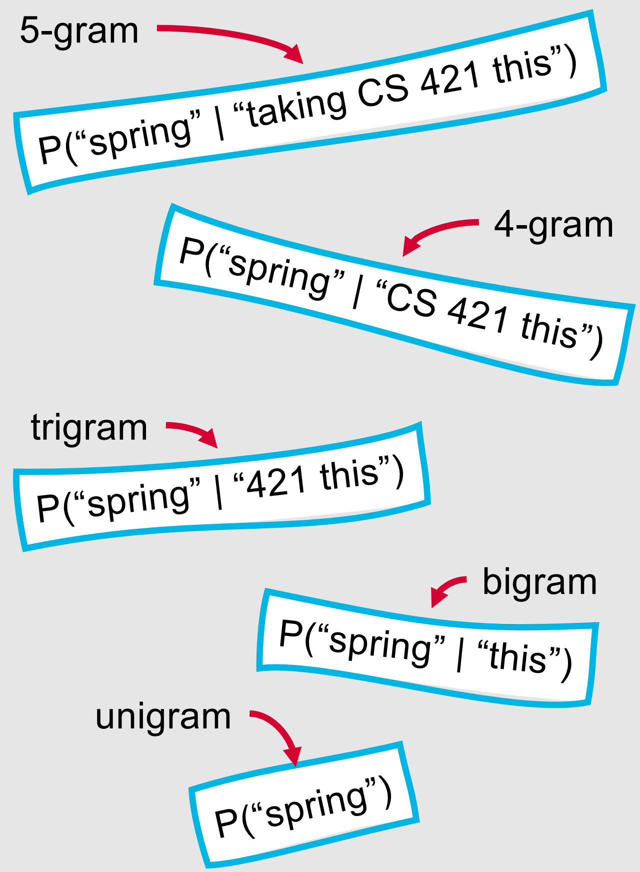
N-grams can be words, characters, or any other type of item in your language.

N-grams are interesting!

N-grams are interesting!

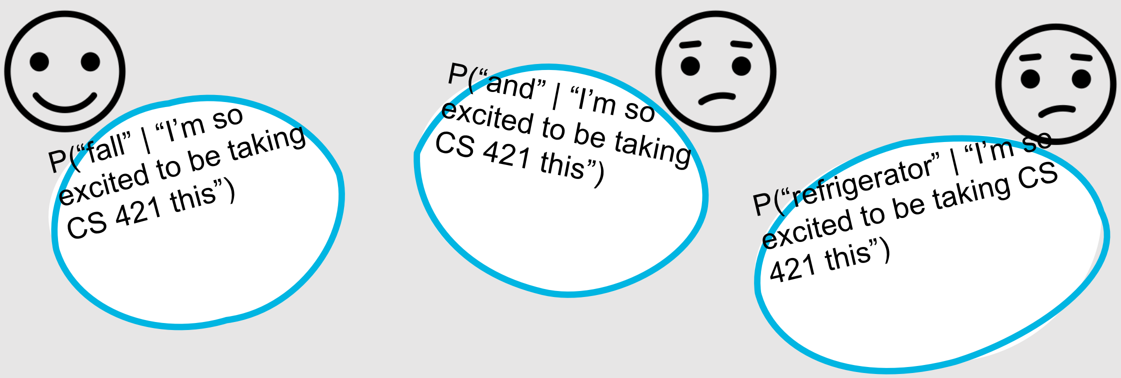
Special N-Grams

- Most higher-order (n>3) n-grams are simply referred to using the value of *n*
  - 4-gram
  - 5-gram
- However, lower-order n-grams are often referred to using special terms:
  - Unigram (1-gram)
  - Bigram (2-gram)
  - Trigram (3-gram)



N-Gram Language Models

- Goal: Predict P(word|history)
  - P(“spring” | “I’m so excited to be taking CS 421 this”)



Probabilities for n-gram language models come from corpus frequencies.

- Intuition:
  1. Take a large corpus
  2. Count the number of times you see the history
  3. Count the number of times the specified word follows the history

P(“spring” | “I’m so excited to be taking CS 421 this”)

= C(“I’m so excited to be taking CS 421 this spring”) / C(“I’m so excited to be taking CS 421 this”)

However, we don’t necessarily want to consider our *entire* history.

- What if our history contains uncommon words?
- What if we have limited computing resources?

P(“spring” | “I’m so excited to be taking **Natalie Parde’s** CS 421 this”)

Out of all possible 11-word sequences on the web, how many are “I’m so excited to be taking Natalie Parde’s CS 421 this”?

Better way of estimating P(word|history)

- Instead of computing the probability of a word given its entire history, **approximate the history using the most recent few words**.
- We do this using fixed-length **n-grams**.

P("spring" | "taking CS 421 this")

P("spring" | "CS 421 this")

P("spring" | "421 this")

P("spring" | "this")

N-gram models follow the **Markov assumption**.

- We can predict the probability of some future unit without looking too far into the past
  - **Bigram language model**: Probability of a word depends only on the previous word
  - **Trigram language model**: Probability of a word depends only on the two previous words
  - **N-gram language model**: Probability of a word depends only on the *n*-1 previous words

More formally….

- $P\left(w_k \middle| w_1^{k-1}\right) \approx P(w_k | w_{k-N+1}^{k-1})$
- We can then multiply these individual word probabilities together to get the probability of a word sequence
  - $P(w_1^n) \approx \prod_{k=1}^{n} P(w_k | w_{k-N+1}^{k-1})$

P("Summer break is already over?")

P("over?" | "already") * P("already" | "is") * P("is" | "break") * P("break" | "Summer")

To compute n-gram probabilities, we can use maximum likelihood estimation.

- **Maximum Likelihood Estimation (MLE):** N-gram frequency counts, normalized to a 0-1 range
  - $P(w_n \mid w_{n-1})$ =
    - # of occurrences of the bigram $w_{n-1}\ w_n$, divided by
    - # of occurrences of the unigram w_{n-1}

Example: Maximum Likelihood Estimation

I am cold.

You are cold.

Everyone is cold.

This is Chicago.

Example: Maximum Likelihood Estimation

I am cold. → <s> I am cold. </s>

You are cold. → <s> You are cold. </s>

Everyone is cold. → <s> Everyone is cold. </s>

This is Chicago. → <s> This is Chicago. </s>

Example: Maximum Likelihood Estimation

I am cold. → <s> I am cold. </s>

You are cold. → <s> You are cold. </s>

Everyone is cold. → <s> Everyone is cold. </s>

This is Chicago. → <s> This is Chicago. </s>

Bigram	Frequency
<s> I	1
I am	1
am cold.	1
cold. </s>	3
…	…
is Chicago.	1
Chicago. </s>	1

Example: Maximum Likelihood Estimation

I am cold. → <s> I am cold. </s>

You are cold. → <s> You are cold. </s>

Everyone is cold. → <s> Everyone is cold. </s>

This is Chicago. → <s> This is Chicago. </s>

Bigram	Freq.
<s> I	1
I am	1
am cold.	1
cold. </s>	3
…	…
is Chicago.	1
Chicago. </s>	1

Unigram	Freq.
<s>	4
I	1
am	1
cold.	3
…	…
Chicago.	1
</s>	4

Example: Maximum Likelihood Estimation

I am cold. → <s> I am cold. </s>

You are cold. → <s> You are cold. </s>

Everyone is cold. → <s> Everyone is cold. </s>

This is Chicago. → <s> This is Chicago. </s>

Bigram	Freq.
<s> I	1
I am	1
am cold.	1
cold. </s>	3
…	…
is Chicago.	1
Chicago. </s>	1

Unigram	Freq.
<s>	4
I	1
am	1
cold.	3
…	…
Chicago.	1
</s>	4

P("I" | "<s>") = C("<s> I") / C("<s>") = 1 / 4 = 0.25

Example: Maximum Likelihood Estimation

I am cold. → <s> I am cold. </s>

You are cold. → <s> You are cold. </s>

Everyone is cold. → <s> Everyone is cold. </s>

This is Chicago. → <s> This is Chicago. </s>

Bigram	Freq.
<s> I	1
I am	1
am cold.	1
cold. </s>	3
…	…
is Chicago.	1
Chicago. </s>	1

Unigram	Freq.
<s>	4
I	1
am	1
cold.	3
…	…
Chicago.	1
</s>	4

P("I" | "<s>") = C("<s> I") / C("<s>") = 1 / 4 = 0.25

P("</s>" | "cold.") = C("cold. </s>") / C("cold.") = 3 / 3 = 1.00

Example: Maximum Likelihood Estimation

I am cold. → <s> I am cold. </s>

You are cold. → <s> You are cold. </s>

Everyone is cold. → <s> Everyone is cold. </s>

This is Chicago. → <s> This is Chicago. </s>

Bigram	Freq.
<s> I	1
I am	1
am cold.	1
cold. </s>	3
…	…
is Chicago.	1
Chicago. </s>	1

Unigram	Freq.
<s>	4
I	1
am	1
cold.	3
…	…
Chicago.	1
</s>	4

P("I" | "<s>") = C("<s> I") / C("<s>") = 1 / 4 = 0.25

P("</s>" | "cold.") = C("cold. </s>") / C("cold.") = 3 / 3 = 1.00

We can learn a lot of useful things from n-gram statistics!

- Syntactic information
  - Do nouns often follow verbs?
  - Do verbs usually follow specific unigrams?
- Task-relevant information
  - Is it likely that virtual assistants will hear the word “I” in a user’s input?
- Cultural or sociological information
  - What phrases are common in articles written by university professors? (How does that vary from phrases in articles written by undergraduate students?)

Which type of n-gram is best?

- In general, the highest-order value of *n* that your data can support
- Sparsity increases with order, and sparse feature vectors are not very useful when training statistical models
- Make sure that your dataset is large enough to handle your selected n-gram size
- We can usually determine this by running experiments on the same data with different n-gram sizes and figuring out which size leads to the best results
- For a deep dive into statistical power in NLP experiments, check out the following paper:
    - *With Little Power Comes Great Responsibility*, by Dallas Card et al.: https://aclanthology.org/2020.emnlp-main.745/

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We've learned how to build n-gram language models, but how do we evaluate them?

- Two types of evaluation paradigms:
  - Extrinsic
  - Intrinsic
- **Extrinsic evaluation:** Embed the language model in an application, and compute changes in task performance
- **Intrinsic evaluation:** Measure the quality of the model, independent of any application

Perplexity

- Intrinsic evaluation metric for language models
- Perplexity (PP) of a language model on a test set is the **inverse probability of the test set**, normalized by the number of words in the test set

More formally….

- $PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \ldots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i | w_1 \ldots w_{i-1})}}$
  - Where *W* is a test set containing words w_1, w_2, …, w_n
  - History size depends on n-gram size
    - $P(w_i | w_{i-1})$ vs $P(w_i | w_{i-2} w_{i-1})$, etc.
- Higher conditional probability of a word sequence → lower perplexity
  - Minimizing perplexity = maximizing test set probability according to the language model

Example: Perplexity

Training Set

Word	Frequency
CS	10
421	10
Statistical	10
Natural	10
Language	10
Processing	10
University	10
of	10
Illinois	10
Chicago	10

Example: Perplexity

Training Set

Word	Frequency
CS	10
421	10
Statistical	10
Natural	10
Language	10
Processing	10
University	10
of	10
Illinois	10
Chicago	10

Test String

CS 421 Statistical Natural Language Processing University of Illinois Chicago

Example: Perplexity

Training Set

Word	Frequency
CS	10
421	10
Statistical	10
Natural	10
Language	10
Processing	10
University	10
of	10
Illinois	10
Chicago	10

Test String

CS 421 Statistical Natural Language Processing University of Illinois Chicago

$$PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \ldots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i | w_1 \ldots w_{i-1})}}$$

Example: Perplexity

Training Set

Word	Frequency
CS	10
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Statistical	10
Natural	10
Language	10
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Test String

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P("CS") = C("CS") / C(<all unigrams>) = 10/100 = 0.1

Example: Perplexity

Training Set

Word	Frequency
CS	10
421	10
Statistical	10
Natural	10
Language	10
Processing	10
University	10
of	10
Illinois	10
Chicago	10

Test String

CS 421 Statistical Natural Language Processing University of Illinois Chicago

$$PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \ldots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i | w_1 \ldots w_{i-1})}}$$

P("CS") = C("CS") / C(<all unigrams>) = 10/100 = 0.1

P("421") = C("421") / C(<all unigrams>) = 10/100 = 0.1

Example: Perplexity

Training Set

Word	Frequency	P(Word)
CS	10	0.1
421	10	0.1
Statistical	10	0.1
Natural	10	0.1
Language	10	0.1
Processing	10	0.1
University	10	0.1
of	10	0.1
Illinois	10	0.1
Chicago	10	0.1

Test String

CS 421 Statistical Natural Language Processing University of Illinois Chicago

$$PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \dots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i|w_1 \dots w_{i-1})}}$$

Example: Perplexity

Training Set

Word	Frequency	P(Word)
CS	10	0.1
421	10	0.1
Statistical	10	0.1
Natural	10	0.1
Language	10	0.1
Processing	10	0.1
University	10	0.1
of	10	0.1
Illinois	10	0.1
Chicago	10	0.1

Test String

CS 421 Statistical Natural Language Processing University of Illinois Chicago

$$PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \ldots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i | w_1 \ldots w_{i-1})}}$$

PP("CS 421 Statistical Natural Language Processing University of Illinois Chicago")

$$= \sqrt[10]{\frac{1}{0.1*0.1*0.1*0.1*0.1*0.1*0.1*0.1*0.1*0.1}} = 10$$

Example: Perplexity

Training Set

Word	Frequency	P(Word)
CS	1	
421	1	
Statistical	1	
Natural	1	
Language	1	
Processing	1	
University	1	
of	1	
Illinois	1	
Chicago	91	

Test String

Illinois Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago

$$PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \ldots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i | w_1 \ldots w_{i-1})}}$$

Example: Perplexity

Training Set

Word	Frequency	P(Word)
CS	1	0.01
421	1	0.01
Statistical	1	0.01
Natural	1	0.01
Language	1	0.01
Processing	1	0.01
University	1	0.01
of	1	0.01
Illinois	1	0.01
Chicago	91	0.91

Test String

Illinois Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago

$$PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \ldots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i | w_1 \ldots w_{i-1})}}$$

Example: Perplexity

Training Set

Word	Frequency	P(Word)
CS	1	0.01
421	1	0.01
Statistical	1	0.01
Natural	1	0.01
Language	1	0.01
Processing	1	0.01
University	1	0.01
of	1	0.01
Illinois	1	0.01
Chicago	91	0.91

Test String

Illinois Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago

$$PP(W) = \sqrt[n]{\frac{1}{P(w_1 w_2 \dots w_n)}} = \sqrt[n]{\prod_{i=1}^{n} \frac{1}{P(w_i | w_1 \dots w_{i-1})}}$$

PP("Illinois Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago Chicago")

$$= \sqrt[10]{\frac{1}{0.01*0.91*0.91*0.91*0.91*0.91*0.91*0.91*0.91*0.91}} = 1.73$$

Perplexity can be used to compare different language models.

Which language model is best?

- Model A: Perplexity = 962
- Model B: Perplexity = 170
- Model C: Perplexity = 109

Perplexity can be used to compare different language models.

Which language model is best?

- Model A: Perplexity = 962
- Model B: Perplexity = 170
- Model C: Perplexity = 109

What kind of perplexity scores are state-of-the-art language models reaching?

- Depends on the dataset
- Recently, as low as:
  - ~10 on WikiText-103: https://paperswithcode.com/sota/language-modelling-on-wikitext-103
  - ~20-30 on Penn Treebank (Word Level): https://paperswithcode.com/sota/language-modelling-on-penn-treebank-word

A cautionary note….

- Improvements in perplexity do not guarantee improvements in task performance!
- However, the two are often correlated (and perplexity is quicker and easier to check)
- Strong language model evaluations also include an extrinsic evaluation component

How can we generate text using an n-gram language model?

1. Select an n-gram randomly from the distribution of all n-grams in the training corpus
2. Randomly select an n-gram from the same distribution, dependent on the previous n-gram
   - If we're using a bigram model and the previous bigram was "CS 421," our next bigram has to start with "421")
3. Repeat until the sentence-final token is reached

N-gram size affects generation output!

Unigram

- To him swallowed confess hear both. Of save on trail for are ay device and rote life have
- Hill he late speaks; or! a more to leg less first you enter

No coherence between words

Bigram

- Why dost stand forth thy canopy, forsooth; he is this palpable hit the King Henry. Live king. Follow.
- What means, sir. I confess she? then all sorts, he is trim, captain.

Minimal local coherence between words

Trigram

- Fly, and will rid me these news of price. Therefore the sadness of parting, as they say, 'tis done.
- This shall forbid it should be branded, if renown made it empty.

More coherence….

4-gram

- King Henry. What! I will go seek the traitor Gloucester. Exeunt some of the watch. A great banquet serv'd in;
- It cannot be but so.

Direct quote from Shakespeare

Why were we generating verbatim Shakespeare text with a 4-gram language model?

- The corpus of all Shakespearean text is relatively small (by modern NLP standards)
- This means higher-order n-gram matrices are sparse:
  - Only five possible continuations for “It cannot be but” (“that,” “I,” “he,” “thou,” and “so”)
    - Probability for all other continuations is assumed to be zero

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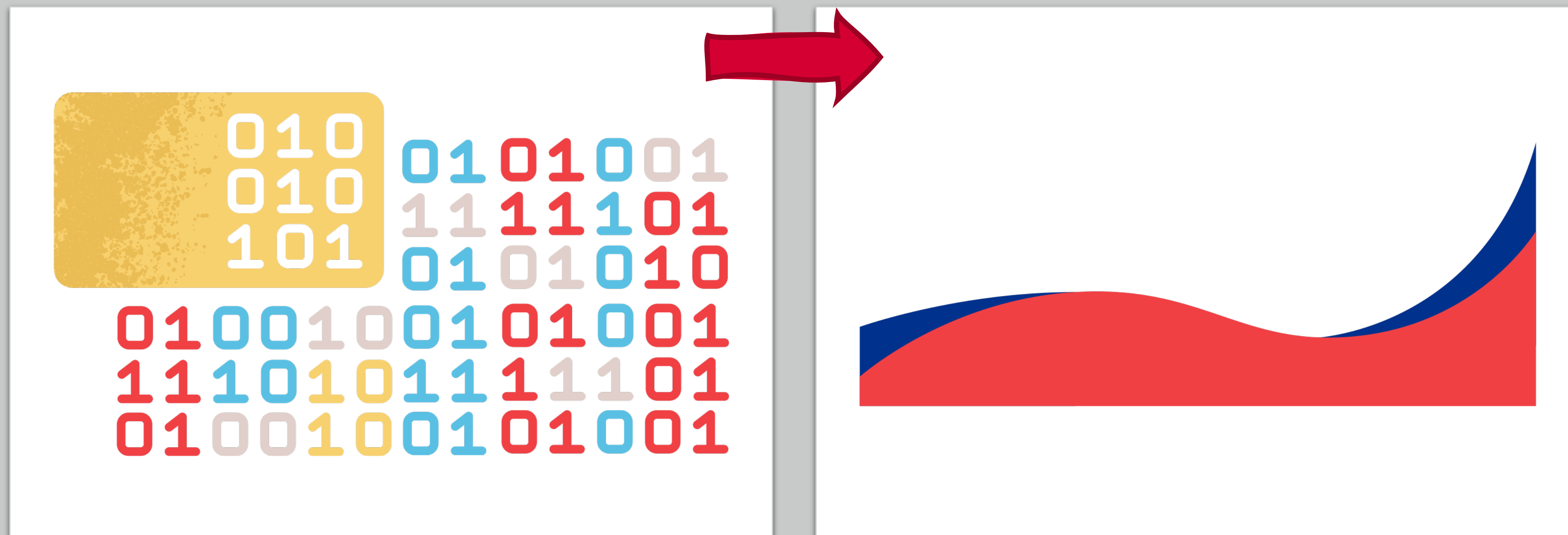
“Zero” probabilities create challenges for language models.

- Zero probabilities occur in two different scenarios:
  - **Unknown words** (**out-of-vocabulary** words)
  - Known words in **unseen contexts**
- Language is varied and often unpredictable—few combinations are *truly* impossible
- Zero probabilities also interfere with perplexity calculations

Modeling Unknown Words

- Add a pseudoword **<UNK>** to the vocabulary
- Then….
  - Option A:
    - Choose a fixed words list
    - Convert any words not in that list to <UNK>
    - Estimate the probabilities for <UNK> like any other word
  - Option B:
    - Replace all words occurring fewer than n times with <UNK>
    - Estimate the probabilities for <UNK> like any other word
  - Option C:
    - Replace the first occurrence of each word with <UNK>
    - Estimate the probabilities for <UNK> like any other word
- Beware: If <UNK> ends up with a high probability (e.g., because you have a small vocabulary), your language model will have artificially lower perplexity!
  - Make sure to compare to other language models using the same vocabulary to avoid gaming this metric

We can handle known words in previously unseen contexts by applying smoothing techniques.



Smoothing

- Taking a bit of the probability mass from more frequent events and giving it to unseen events.
  - Sometimes also called “discounting”
- Many different smoothing techniques:
  - Laplace (add-one)
  - Add-k
  - Stupid backoff
  - Kneser-Ney

Bigram	Frequency
CS 421	8
CS 590	5
CS 594	2
CS 521	0 😢

Bigram	Frequency
CS 421	7
CS 590	5
CS 594	2
CS 521	1 🥰

Laplace Smoothing

- Add one to all n-gram counts before they are normalized into probabilities
- Not the highest-performing technique, but a useful baseline
  - Practical method for other text classification tasks
- $P(w_i) = \frac{c_i}{N} \rightarrow P_{\text{Laplace}}(w_i) = \frac{c_i+1}{N+V}$

Example: Laplace Smoothing

Corpus Statistics:

Unigram	Frequency
Chicago	4
is	8
cold	6
hot	0

Bigram	Frequency
Chicago is	2
is cold	4
is hot	0
…	0

Example: Laplace Smoothing

Corpus Statistics:

Unigram	Frequency
Chicago	4
is	8
cold	6
hot	0

Bigram	Frequency
Chicago is	2
is cold	4
is hot	0
…	0

$$P(w_i) = \frac{c_i}{N}$$

Unigram	Probability
Chicago	$\frac{4}{18} = 0.22$
is	$\frac{8}{18} = 0.44$
cold	$\frac{6}{18} = 0.33$
hot	$\frac{0}{18} = 0.00$

Bigram	Probability
Chicago is	
is cold	
is hot	

Example: Laplace Smoothing

Corpus Statistics:

Unigram	Frequency
Chicago	4
is	8
cold	6
hot	0

Bigram	Frequency
Chicago is	2
is cold	4
is hot	0
…	0

$$P(w_i) = \frac{c_i}{N}$$

Unigram	Probability
Chicago	$\frac{4}{18} = 0.22$
is	$\frac{8}{18} = 0.44$
cold	$\frac{6}{18} = 0.33$
hot	$\frac{0}{18} = 0.00$

Bigram	Probability
Chicago is	$\frac{2}{4} = 0.50$
is cold	$\frac{4}{8} = 0.50$
is hot	$\frac{0}{8} = 0.00$

Example: Laplace Smoothing

Corpus Statistics:

Unigram	Frequency
Chicago	4
is	8
cold	6
hot	0

Bigram	Frequency
Chicago is	2
is cold	4
is hot	0
…	0

$P(w_i) = \frac{c_i}{N} \rightarrow P_{\text{Laplace}}(w_i) = \frac{c_i+1}{N+V}$

Unigram	Probability
Chicago	
is	
cold	
hot	

Bigram	Probability
Chicago is	
is cold	
is hot	

Example: Laplace Smoothing

Corpus Statistics:

Unigram	Frequency
Chicago	4+1
is	8+1
cold	6+1
hot	0+1

Bigram	Frequency
Chicago is	2+1
is cold	4+1
is hot	0+1
…	0+1

$$P(w_i) = \frac{c_i}{N} \rightarrow P_{\text{Laplace}}(w_i) = \frac{c_i+1}{N+V}$$

Unigram	Probability
Chicago	
is	
cold	
hot	

Bigram	Probability
Chicago is	
is cold	
is hot	

Example: Laplace Smoothing

Corpus Statistics:

Unigram	Frequency
Chicago	4+1
is	8+1
cold	6+1
hot	0+1

Bigram	Frequency
Chicago is	2+1
is cold	4+1
is hot	0+1
…	0+1

$$P(w_i) = \frac{c_i}{N} \rightarrow P_{\text{Laplace}}(w_i) = \frac{c_i+1}{N+V}$$

Unigram	Probability
Chicago	$\frac{5}{22} = 0.23$
is	$\frac{9}{22} = 0.41$
cold	$\frac{7}{22} = 0.32$
hot	$\frac{1}{22} = 0.05$

Bigram	Probability
Chicago is	
is cold	
is hot	

Example: Laplace Smoothing

Corpus Statistics:

Bigram	Frequency
Chicago Chicago	0+1
Chicago is	2+1
Chicago cold	0+1
Chicago hot	0+1

$P(w_i) = \frac{c_i}{N} \rightarrow P_{\text{Laplace}}(w_i) = \frac{c_i+1}{N+V}$

Unigram	Frequency
Chicago	4
is	8
cold	6
hot	0

Bigram	Frequency
Chicago is	2+1
is cold	4+1
is hot	0+1
…	0+1

Unigram	Probability
Chicago	$\frac{5}{22} = 0.23$
is	$\frac{9}{22} = 0.41$
cold	$\frac{7}{22} = 0.32$
hot	$\frac{1}{22} = 0.05$

Bigram	Probability
Chicago is	$\frac{3}{4+4} = \frac{3}{8} = 0.38$
is cold	$\frac{5}{8+4} = \frac{5}{12} = 0.42$
is hot	$\frac{1}{8+4} = \frac{1}{12} = 0.08$

Probabilities: Before and After

Bigram	Probability
Chicago is	$\frac{2}{4} = 0.50$
is cold	$\frac{4}{8} = 0.50$
is hot	$\frac{0}{8} = 0.00$

Bigram	Probability
Chicago is	$\frac{3}{8} = 0.38$
is cold	$\frac{5}{12} = 0.42$
is hot	$\frac{1}{12} = 0.08$

Add-K Smoothing

- Moves a bit less of the probability mass from seen to unseen events
- Rather than adding one to each count, add a fractional count (e.g., 0.5 or 0.01)
  - $P(w_i) = \frac{c_i}{N} \rightarrow P_{\mathrm{Add-K}}(w_i) = \frac{c_i+k}{N+kV}$
  - $P(w_n|w_{n-1}) = \frac{c(w_{n-1}w_n)}{c(w_{n-1})} \rightarrow$
    $P_{\mathrm{Add-K}}(w_n|w_{n-1}) = \frac{c(w_{n-1}w_n)+k}{c(w_{n-1})+kV}$
- This smoothing technique is more customizable: the value *k* can be optimized on a portion of the dataset

Add-K smoothing is useful for some tasks, but still tends to be suboptimal for language modeling.

- Other smoothing techniques?
  - **Backoff:** Use the specified n-gram size to estimate probability if its count is greater than 0; otherwise, *backoff* to a smaller-size n-gram until you reach a size with non-zero counts
  - **Interpolation:** Mix the probability estimates from multiple n-gram sizes, weighing and combining the n-gram counts

Katz Backoff

- Incorporate a function α to distribute probability mass to lower-order n-grams
- Rely on a discounted probability P* if the n-gram has non-zero counts
- Otherwise, recursively back off to the Katz probability for the (n-1)-gram
- $P_{BO}(w_n|w_{n-N+1}^{n-1}) = \begin{cases} P^*(w_n|w_{n-N+1}^{n-1}), & \text{if } c(w_{n-N+1}^{n}) > 0 \\ \alpha(w_{n-N+1}^{n-1})P_{BO}(w_n|w_{n-N+2}^{n-1}), & \text{otherwise} \end{cases}$

Interpolation

N	Weight
3	0.5
2	0.4
1	0.1

N-Gram	Probability	Value
I ❤️ 421	P(421 | I ❤️)	0.7
❤️ 421	P(421 | ❤️)	0.5
421	P(421)	0.2

$\boxed{0.5 * 0.7 + 0.4 * 0.5 + 0.1 * 0.2 = 0.57}$

- **Linear interpolation**
  - $P'(w_n|w_{n-2}w_{n-1}) = \lambda_1 P(w_n|w_{n-2}w_{n-1}) + \lambda_2 P(w_n|w_{n-1}) + \lambda_3 P(w_n)$
    - Where $\sum_i \lambda_i = 1$
- **Conditional interpolation**
  - $P'(w_n|w_{n-2}w_{n-1}) = \lambda_1(w_{n-2}^{n-1})P(w_n|w_{n-2}w_{n-1}) + \lambda_2(w_{n-2}^{n-1})P(w_n|w_{n-1}) + \lambda_3(w_{n-2}^{n-1})P(w_n)$

Context-conditioned weights

N-Gram	Probability	Value	Weight
I ❤️ 421	P(421 | I ❤️)	0.7	0.5
I 🚕 421	P(421 | I 🚕)	0.7	0.1

Kneser-Ney Smoothing

- Objective: Capture the intuition that although some lower-order n-grams are frequent, they are mainly *only frequent in specific contexts*
  - tall nonfat decaf peppermint _______
    - “york” is a more frequent unigram than “mocha” in most datasets, but it’s mainly frequent when it follows the word “new”
- Creates a unigram model that estimates the probability of seeing the word *w* as a novel continuation, in a new unseen context
  - Based on the number of different contexts in which *w* has already appeared

Kneser-Ney Smoothing

$$P_{\mathrm{KN}}(w_i|w_{i-n+1}^{i-1}) = \frac{\max\left(c_{KN}\left(w_{i-n+1}^{i}\right) - d,\ \ 0\right)}{\sum_v c_{KN}\left(w_{i-n+1}^{i-1}v\right)} + \lambda(w_{i-n+1}^{i-1})P_{\mathrm{KN}}(w_i|w_{i-n+2}^{i-1})$$

Kneser-Ney Smoothing

$$P_{\text{KN}}(w_i|w_{i-n+1}^{i-1}) = \frac{\max\left(c_{KN}\left(w_{i-n+1}^{i}\right) - d,\ \ 0\right)}{\sum_v c_{KN}\left(w_{i-n+1}^{i-1}v\right)} + \lambda(w_{i-n+1}^{i-1})P_{\text{KN}}(w_i|w_{i-n+2}^{i-1})$$

Normalizing constant to distribute the probability mass that's been discounted

$$\lambda(w_{i-1}) = \frac{d}{\sum_v C(w_{i-1}v)}|\{w : c(w_{i-1}w) > 0\}|$$

Normalized discount

Number of word types that can follow w_{i-1}

Kneser-Ney Smoothing

$$P_{\mathrm{KN}}(w_i|w_{i-n+1}^{i-1}) = \frac{\max(c_{KN}(w_{i-n+1}^{i}) - d,\ \ 0)}{\sum_v c_{KN}(w_{i-n+1}^{i-1}v)} + \lambda(w_{i-n+1}^{i-1}) P_{\mathrm{KN}}(w_i|w_{i-n+2}^{i-1})$$

Normalizing constant to distribute the probability mass that's been discounted

Regular count for the highest-order n-gram, or the number of unique single word contexts for lower-order n-grams

Kneser-Ney Smoothing

$$P_{\text{KN}}(w_i|w_{i-n+1}^{i-1}) = \frac{\max(c_{KN}(w_{i-n+1}^{i}) - d,\ \ 0)}{\sum_v c_{KN}(w_{i-n+1}^{i-1}v)} + \lambda(w_{i-n+1}^{i-1}) P_{\text{KN}}(w_i|w_{i-n+2}^{i-1})$$

Normalizing constant to distribute the probability mass that's been discounted

Regular count for the highest-order n-gram, or the number of unique single word contexts for lower-order n-grams

Discounted n-gram probability …when the recursion terminates, unigrams are interpolated with the uniform distribution (ε = empty string)

$$P_{KN}(w) = \frac{\max(c_{KN}(w) - d, 0)}{\sum_{w'} c_{KN}(w')} + \lambda(\varepsilon)\frac{1}{V}$$

Stupid Backoff

- Doesn't even try to make the language model a true probability distribution 😌 (so doesn't discount higher-order probabilities)
- If a higher-order n-gram has a zero count, backs off to a lower-order n-gram, weighted by a fixed weight
- $S\left(w_i \middle| w_{i-k+1}^{i-1}\right) = \begin{cases} \frac{c(w_{i-k+1}^{i})}{c(w_{i-k+1}^{i-1})} & \text{if } c\left(w_{i-k+1}^{i}\right) > 0 \\ \lambda S\left(w_i \middle| w_{i-k+2}^{i-1}\right) & \text{otherwise} \end{cases}$
  - Terminates in the unigram, which has the probability:
    - $S(w) = \frac{c(w)}{N}$

Generally, 0.4 works well (Brants et al., 2007)

Summary: Language Modeling with N-Grams

- **N-grams:** Sequences of *n* items (e.g., characters or words)
- **Language models:** Statistical models of language based on observed word or character co-occurrences
- N-gram probabilities can be computed using **maximum likelihood estimation**
- Language models can be **intrinsically evaluated** using **perplexity**
- Unknown words can be handled using **<UNK>** tokens
- Known words in unseen contexts can be handled using **smoothing**

Hidden Markov Models

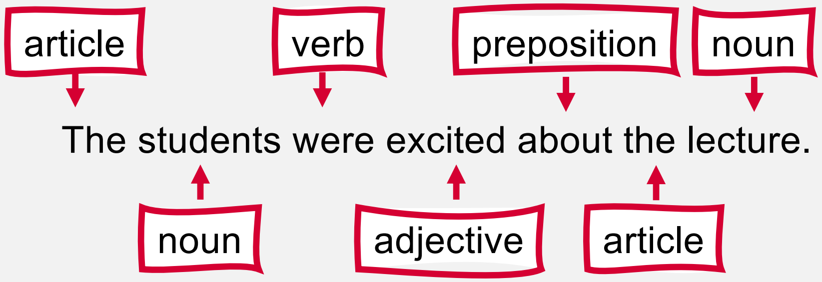
Natalie Parde

UIC CS 421

2

Sequence Labeling

- N-gram language modeling is one way to model sequences of language input
- We can also perform **sequence labeling** by assigning labels to individual tokens or spans of tokens given a longer input string



Sequence Labeling

- Objective: Find the label for the next item, based on the labels of other items in the sequence.

verb determiner

Give me a **break**!

pronoun noun

verb noun

Did the window **break**?

determiner verb

Why perform sequence labeling?

- In document-level text classification, models assume that the individual datapoints being classified are disconnected and independent
- **Many NLP problems do not satisfy this assumption!** Instead, they involve
  - Interconnected, mutually dependent decisions
  - Each of which resolve different ambiguities

Example Sequence Labeling Applications

- Named entity recognition
- Semantic role labeling

person organization

Natalie Parde works at the **University of Illinois at Chicago** and lives in **Chicago, Illinois**.

location

agent source destination

Natalie drove for 15 hours from **Dallas** to **Chicago** in her hail-damaged **Honda Accord**.

instrument

This Week's Topics

Tuesday

- N-gram language modeling
- Evaluating LMs
- Improving n-gram LMs

Thursday

- Hidden Markov Models
- Forward Algorithm
- Viterbi Algorithm
- Forward-Backward Algorithm

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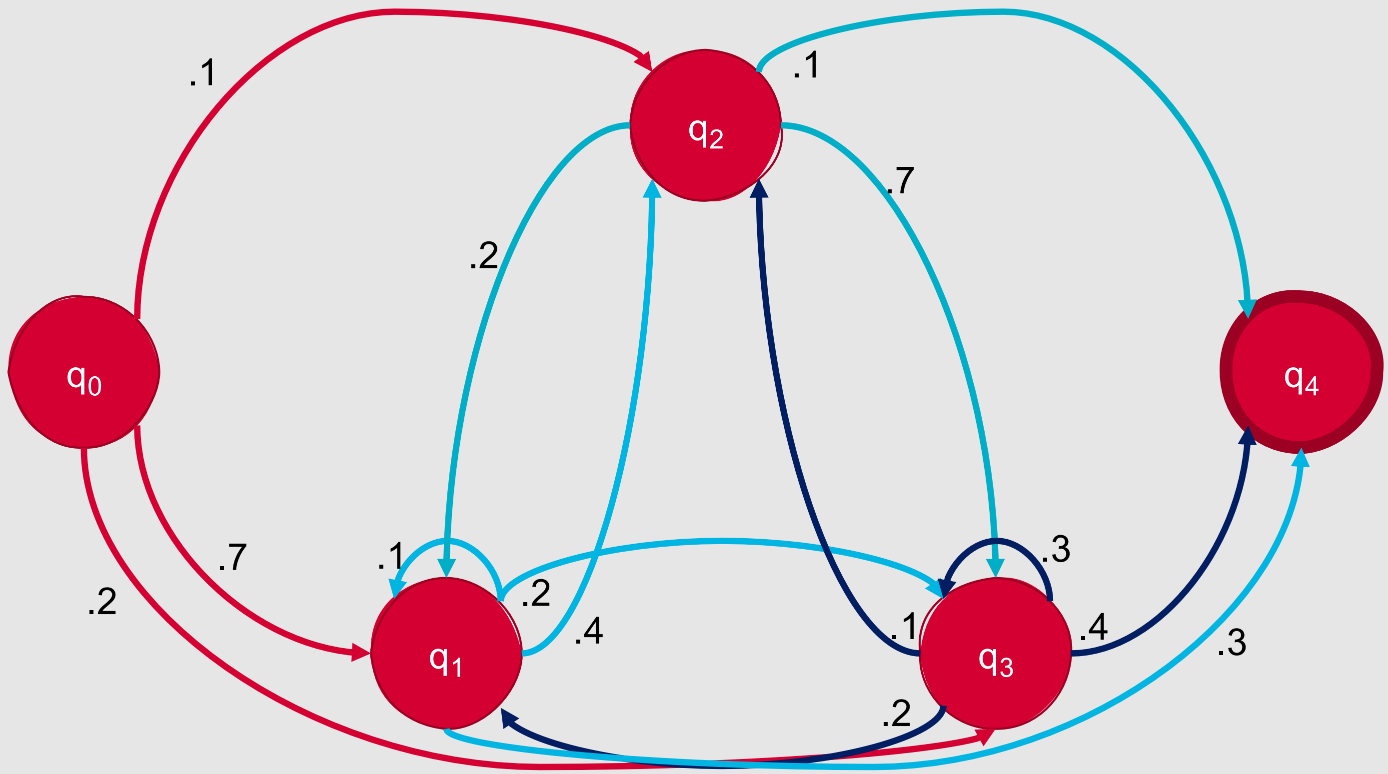
Probabilistic Sequence Models

- We can perform multiple, interdependent classifications to address a greater problem using probabilistic sequence models
- Models that build upon principles used to develop finite state automata are known as **hidden Markov models**
- Hidden Markov models are **probabilistic generative models for sequences** that make predictions based on an underlying set of **hidden states**

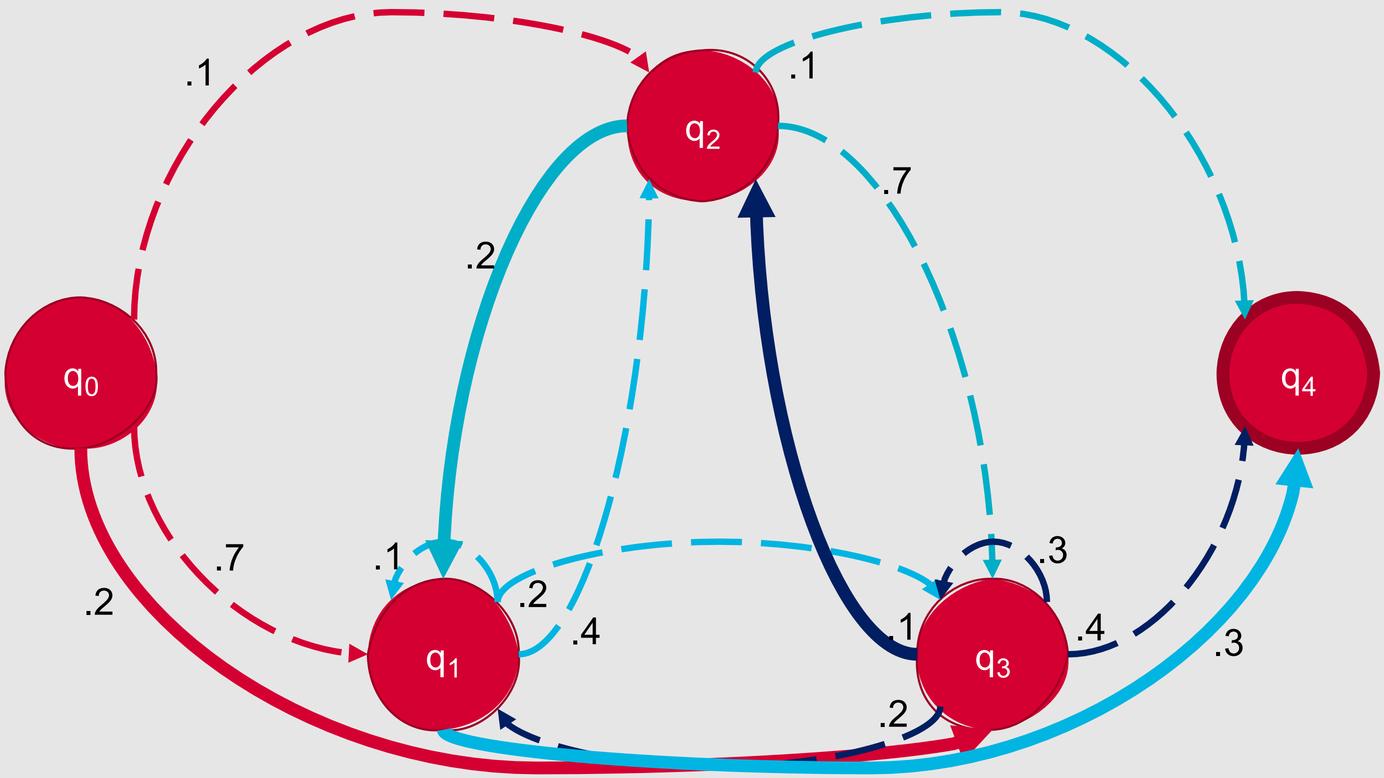
What are Markov Models?

- **Finite state automata with probabilistic state transitions**
- Markov Property: The future is independent of the past, given the present.
  - In other words, the next state only depends on the current state …it is independent of previous history.
- Also referred to as **Markov Chains**

Sample Markov Model



Sample Markov Model



$P(q_3\ q_2\ q_1\ q_4)$
$= .2 * .1 * .2 * .3$
$= .0012$

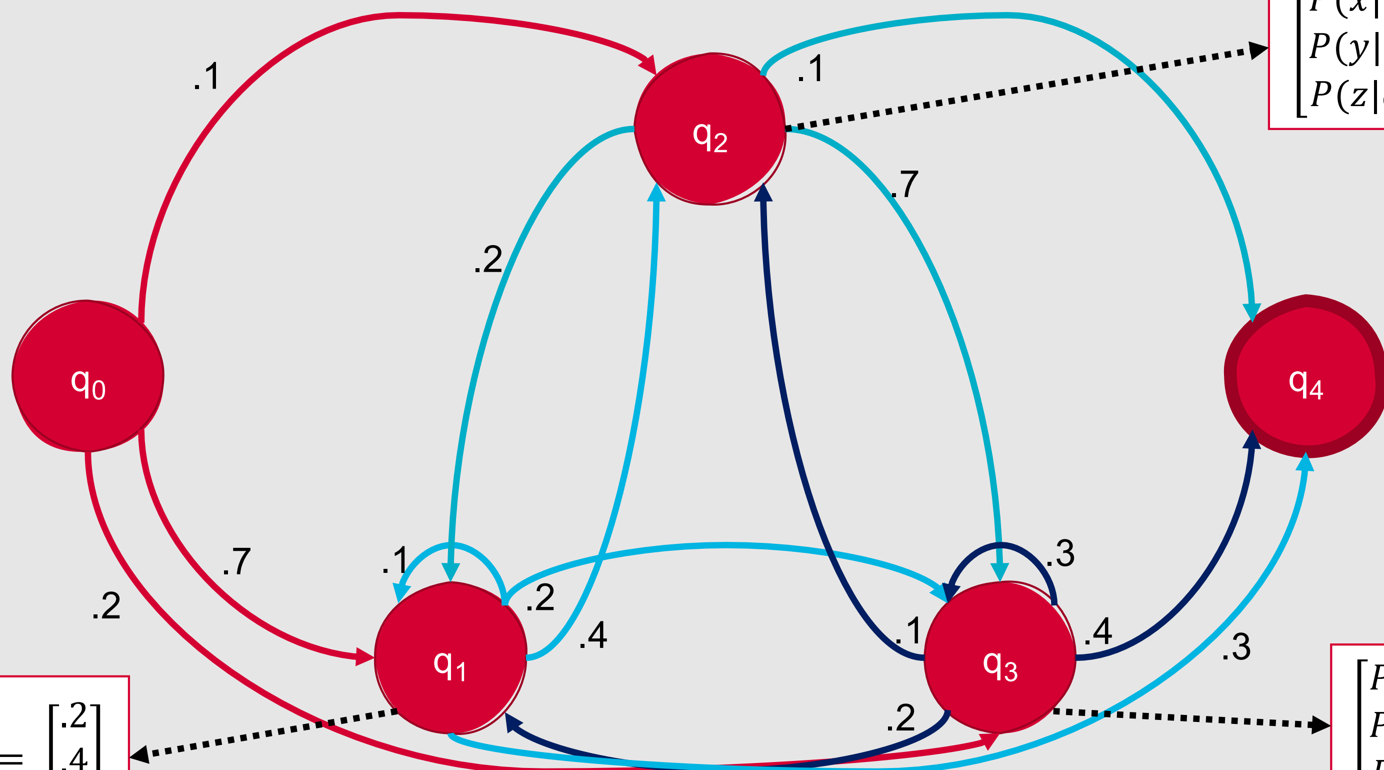
Hidden Markov Models

- Markov models that assume an underlying set of hidden (unobserved) states in which the model can be
- Assume probabilistic transitions between states over time
- Assume probabilistic generation of items (e.g., tokens) from states

Formal Definition

- A Hidden Markov Model can be specified by enumerating the following properties:
  - The set of states, ***Q***
  - A sequence of observation likelihoods, ***B***, also called emission probabilities, each expressing the probability of an observation being generated from a state *i*
  - A start state, q_0, and final state, q_F, that are not associated with observations

Sample Hidden Markov Model



$$\begin{bmatrix} P(x|q_2) \\ P(y|q_2) \\ P(z|q_2) \end{bmatrix} = \begin{bmatrix} .1 \\ .4 \\ .5 \end{bmatrix}$$

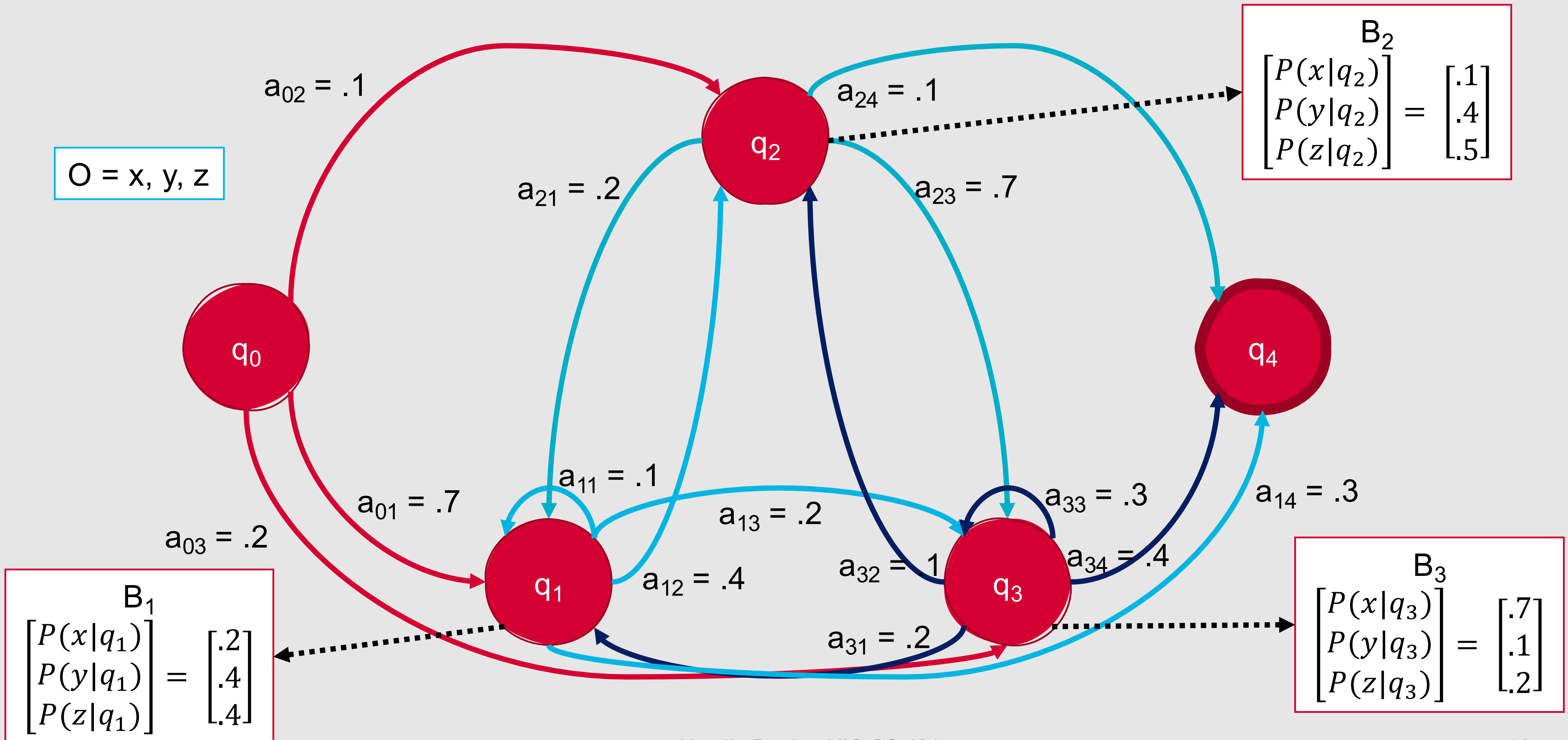
$$\begin{bmatrix} P(x|q_1) \\ P(y|q_1) \\ P(z|q_1) \end{bmatrix} = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}$$

$$\begin{bmatrix} P(x|q_3) \\ P(y|q_3) \\ P(z|q_3) \end{bmatrix} = \begin{bmatrix} .7 \\ .1 \\ .2 \end{bmatrix}$$

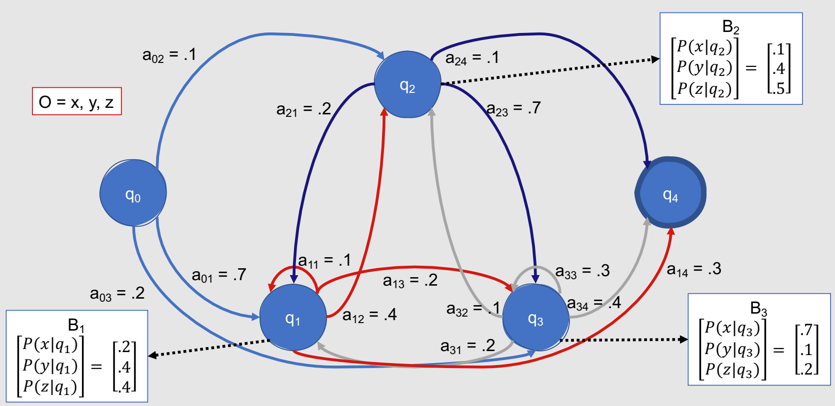
Formal Definition

- A Hidden Markov Model can be specified by enumerating the following properties:
  - The set of states, ***Q***
  - A sequence of observation likelihoods, ***B***, also called emission probabilities, each expressing the probability of an observation o_t being generated from a state *i*
  - A start state, q_0, and final state, q_F, that are not associated with observations, together with transition probabilities out of q_0 and into q_F
  - A transition probability matrix, ***A***, where each a_{ij} represents the probability of moving from state *i* to state *j*, such that $\sum_{j=1}^{n} a_{ij} = 1 \; \forall i$
  - A sequence of T observations, ***O***, each drawn from a vocabulary V = $v_1, v_2, \ldots, v_V$

Sample Hidden Markov Model

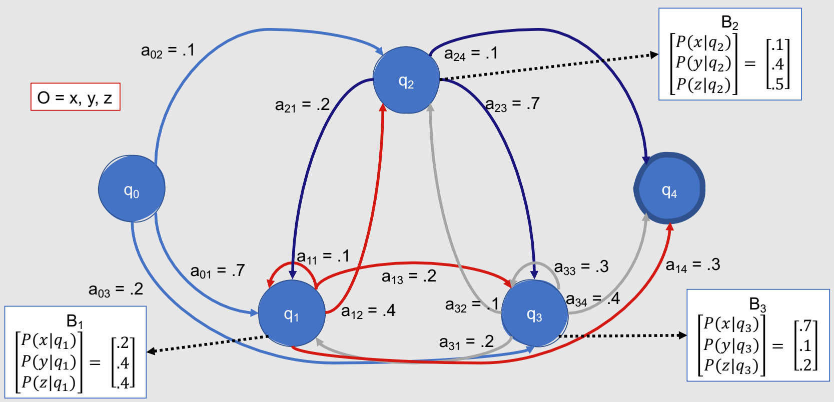


Corresponding Transition Matrix



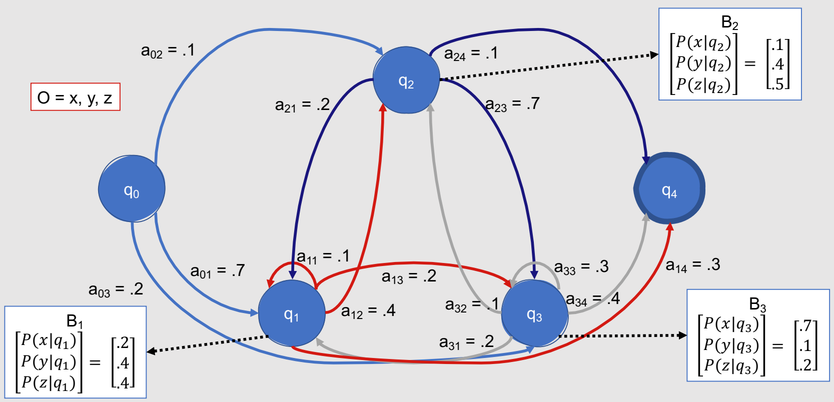
	q0	q1	q2	q3	q4
q0	N/A	.7	.1	.2	N/A
q1					
q2					
q3					
q4					

Corresponding Transition Matrix



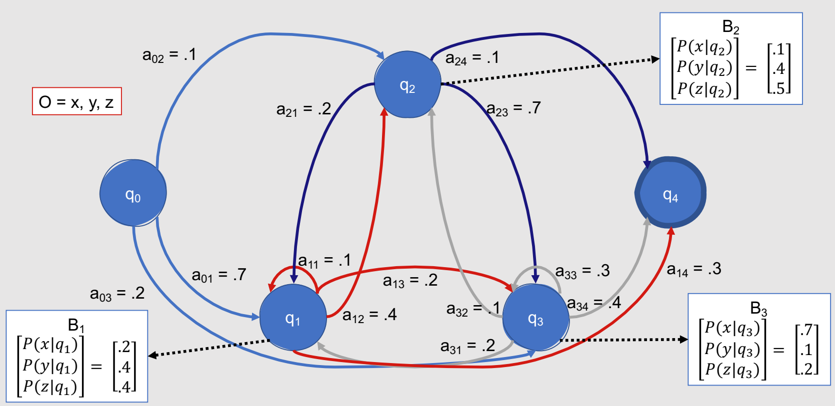
	q0	q1	q2	q3	q4
q0	N/A	.7	.1	.2	N/A
q1	N/A	.1	.4	.2	.3
q2					
q3					
q4					

Corresponding Transition Matrix



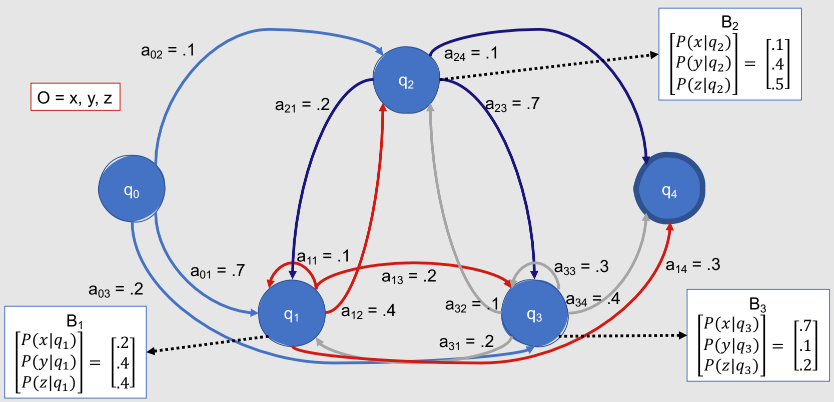
	q0	q1	q2	q3	q4
q0	N/A	.7	.1	.2	N/A
q1	N/A	.1	.4	.2	.3
q2	N/A	.2	N/A	.7	.1
q3					
q4					

Corresponding Transition Matrix



	q0	q1	q2	q3	q4
q0	N/A	.7	.1	.2	N/A
q1	N/A	.1	.4	.2	.3
q2	N/A	.2	N/A	.7	.1
q3	N/A	.2	.1	.3	.4
q4					

Corresponding Transition Matrix



	q0	q1	q2	q3	q4
q0	N/A	.7	.1	.2	N/A
q1	N/A	.1	.4	.2	.3
q2	N/A	.2	N/A	.7	.1
q3	N/A	.2	.1	.3	.4
q4	N/A	N/A	N/A	N/A	N/A

HMMs can also be used for probabilistic text generation!

- More generally, you can use an HMM to generate a sequence of T observations: $O = o_1, o_2, \ldots, o_T$

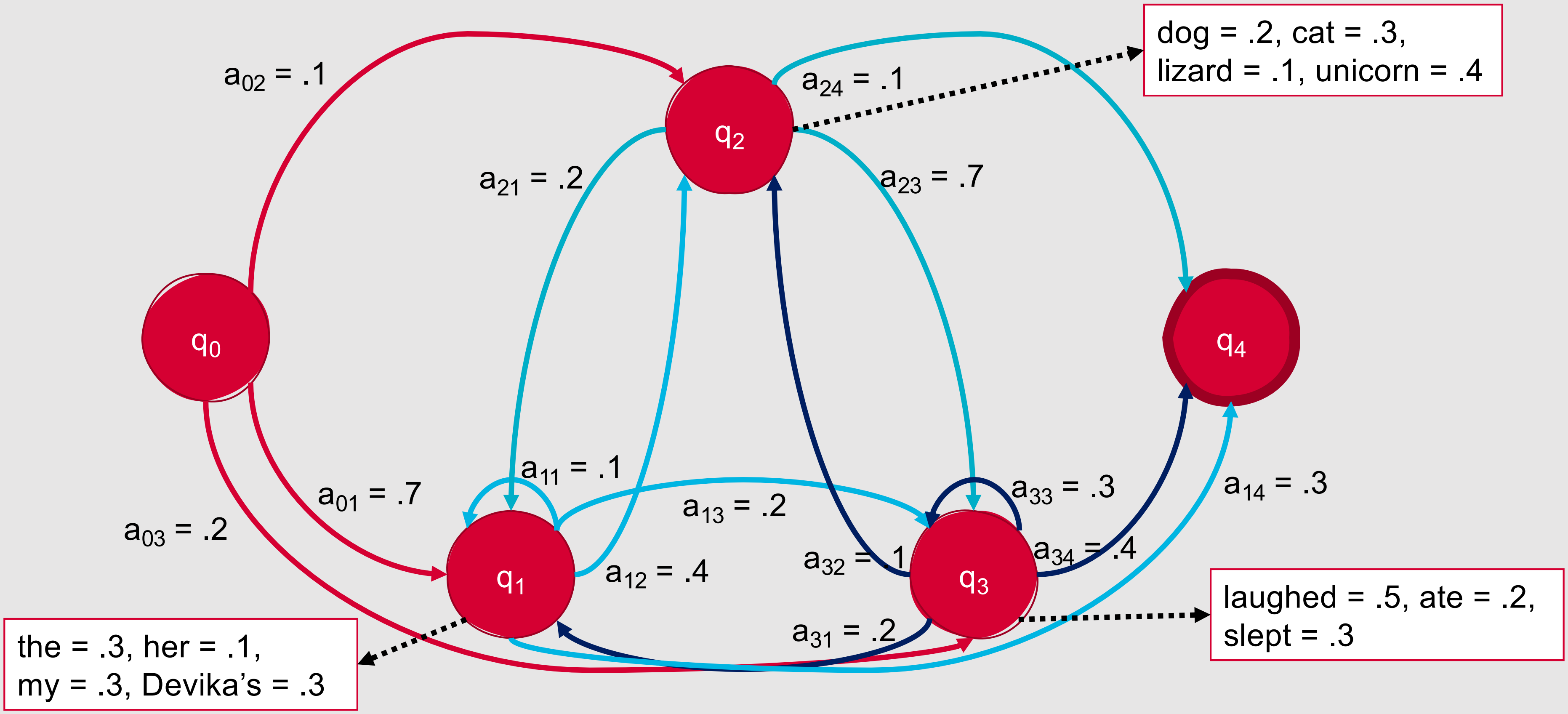
Begin in the start state

For t in [0, …, T]:

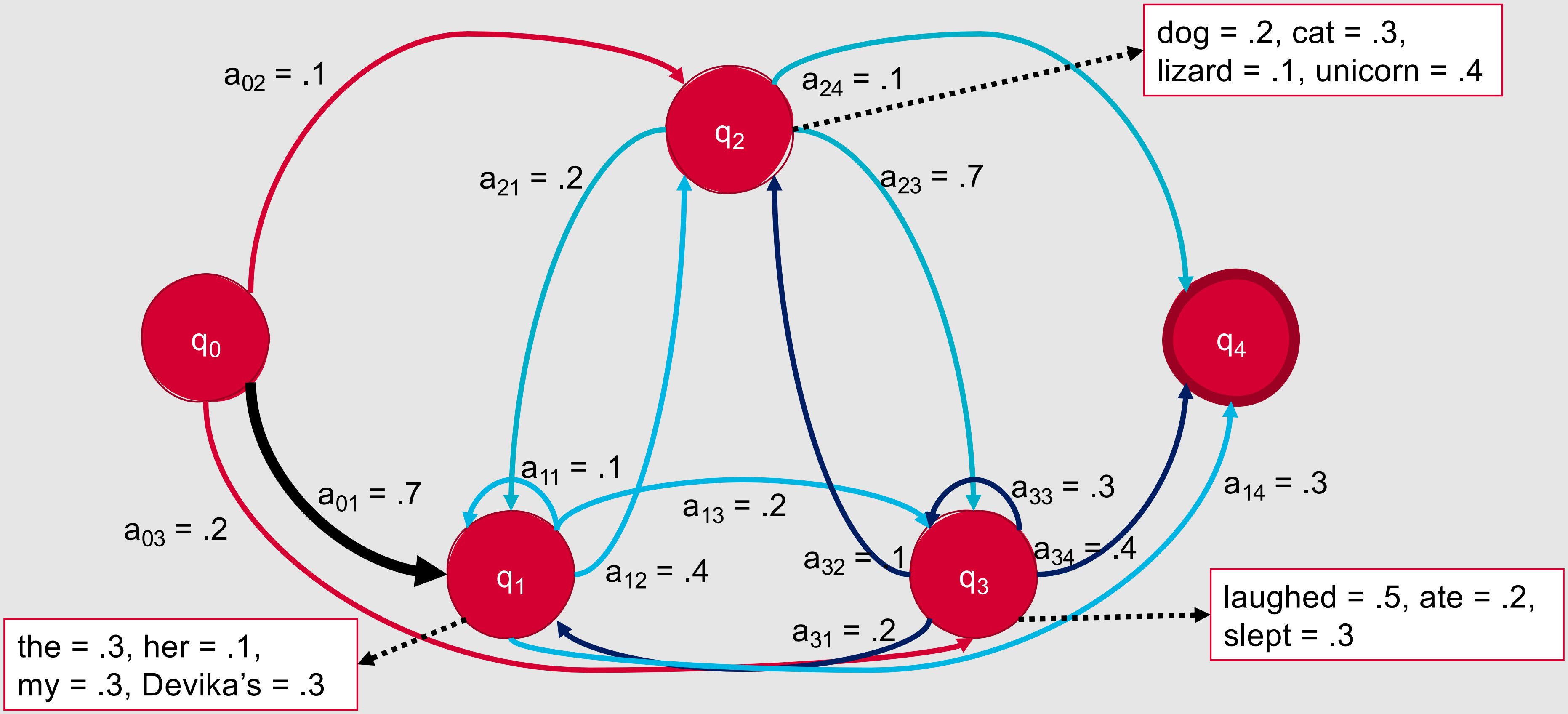
Randomly select a new state based on the transition distribution for the current state

Randomly select an observation from the new state based on the observation distribution for that state

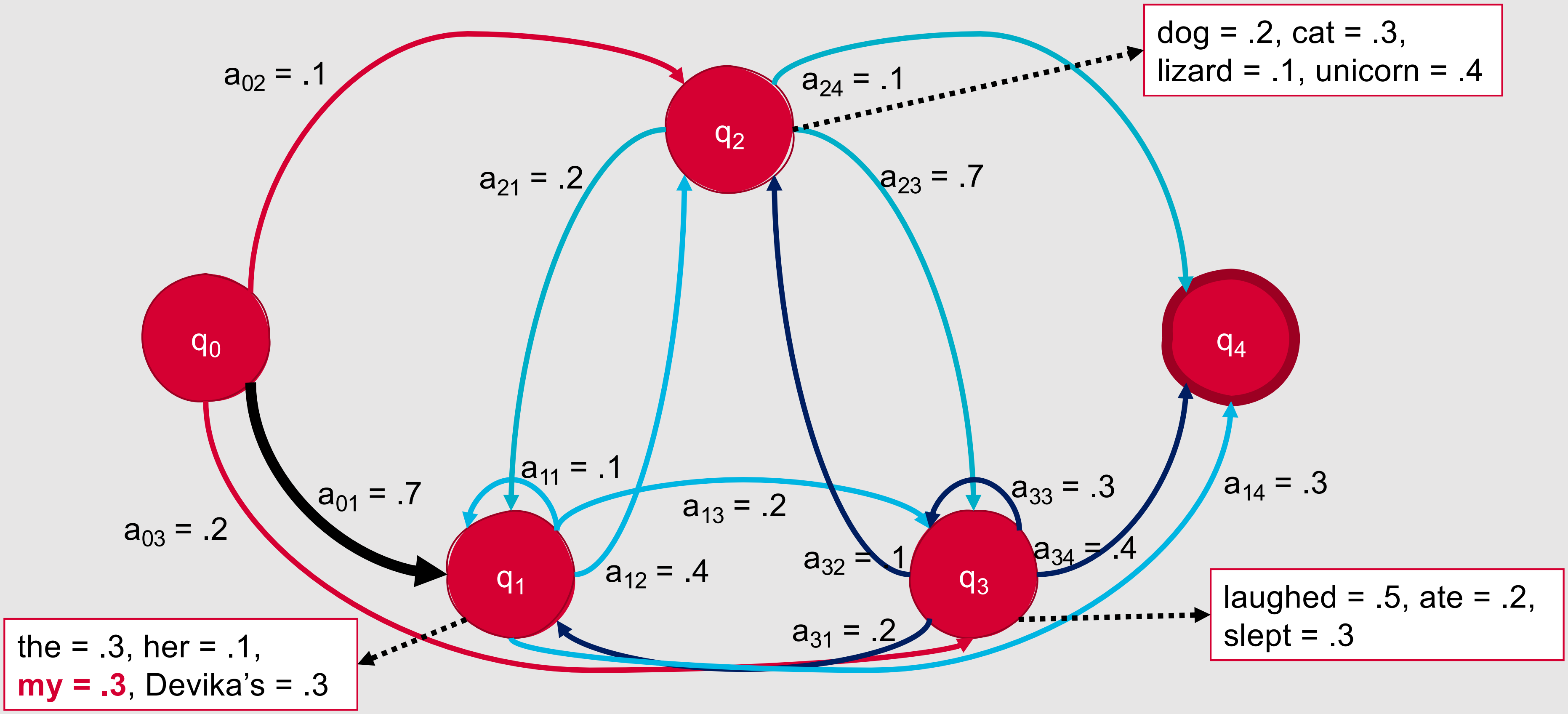
Sample Text Generation



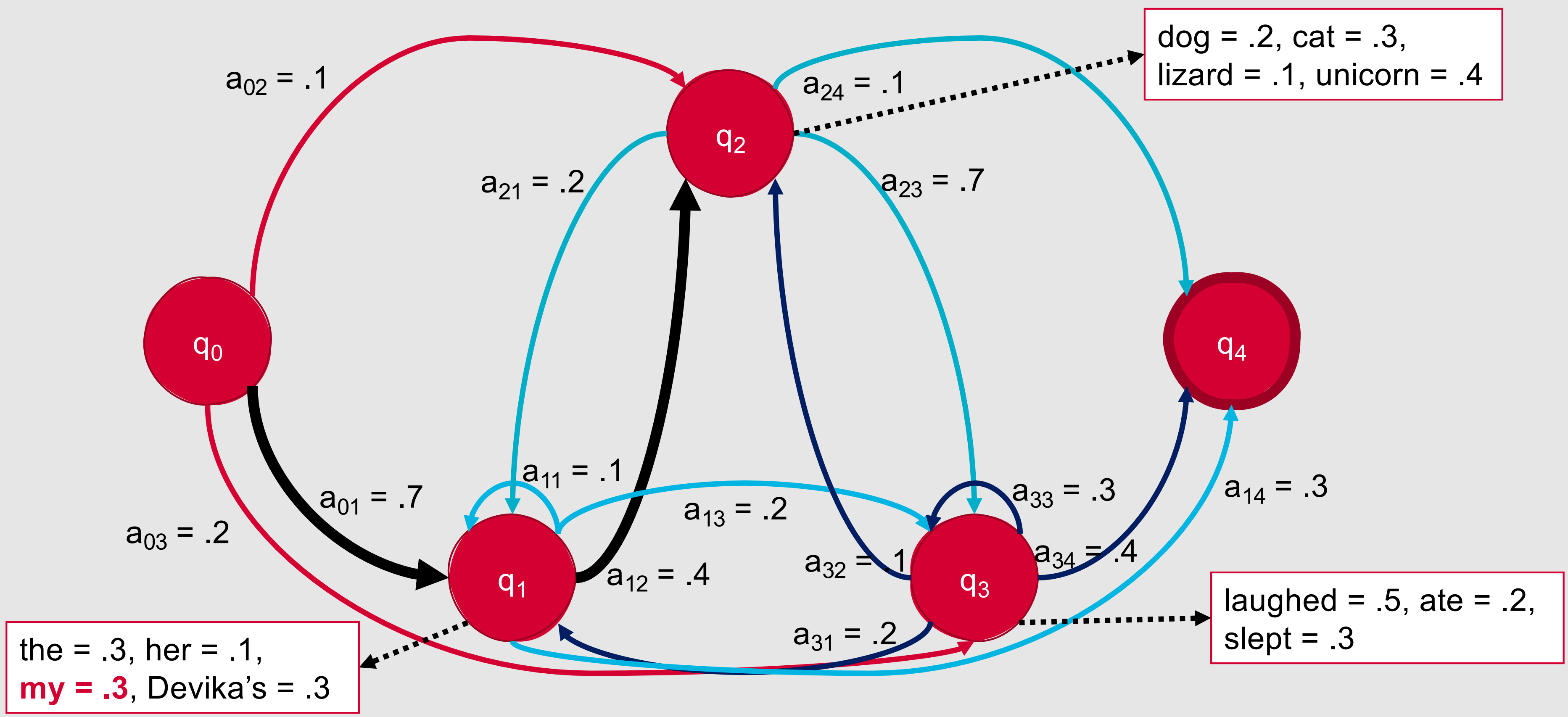
Sample Text Generation



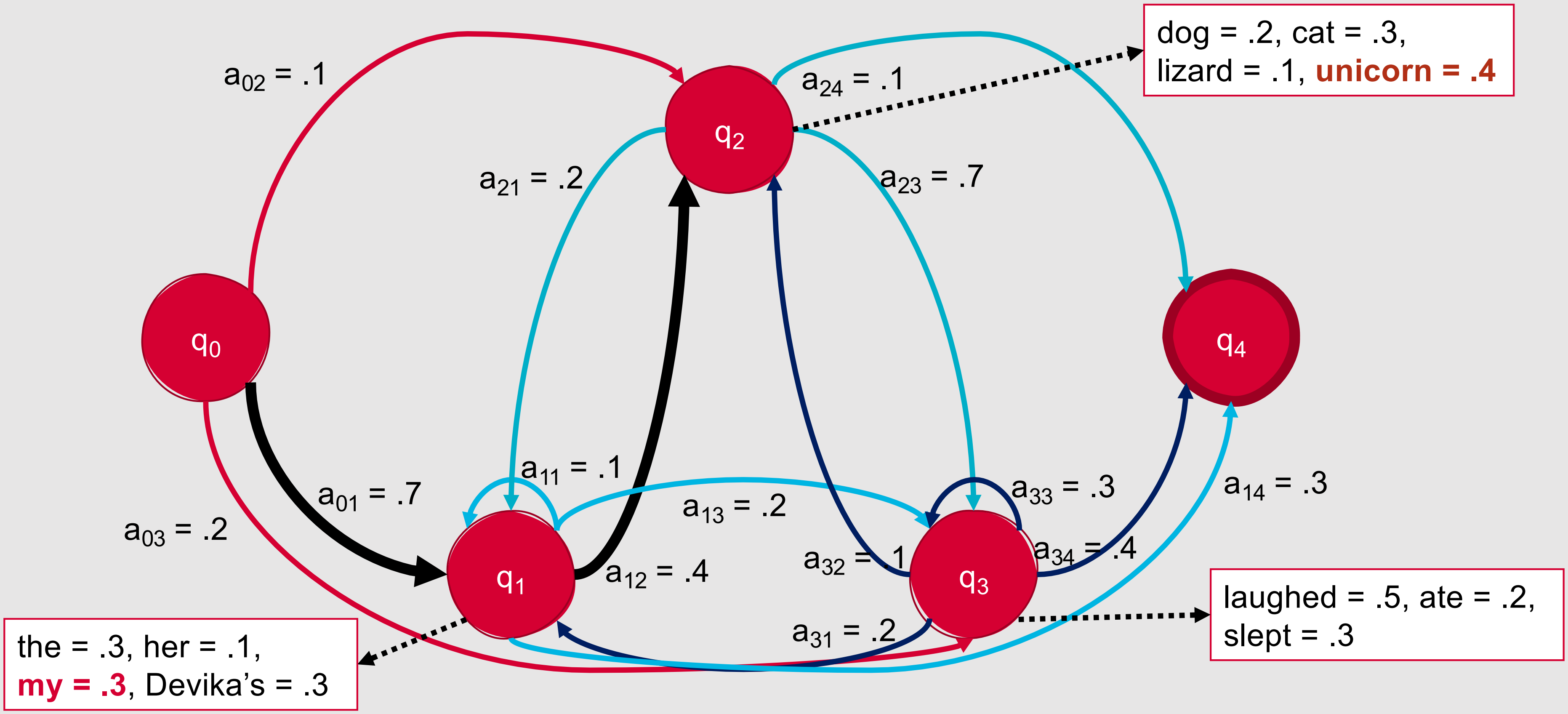
Sample Text Generation



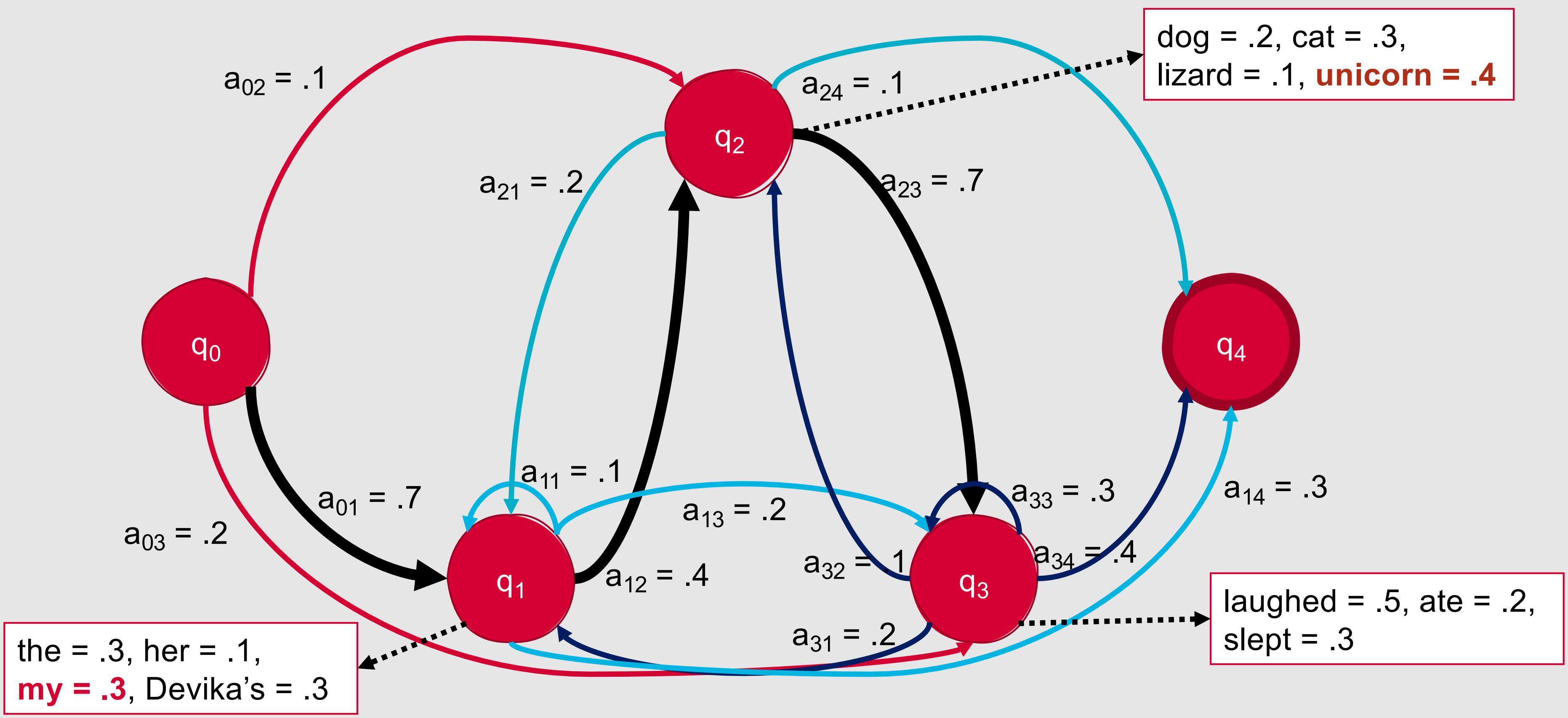
Sample Text Generation



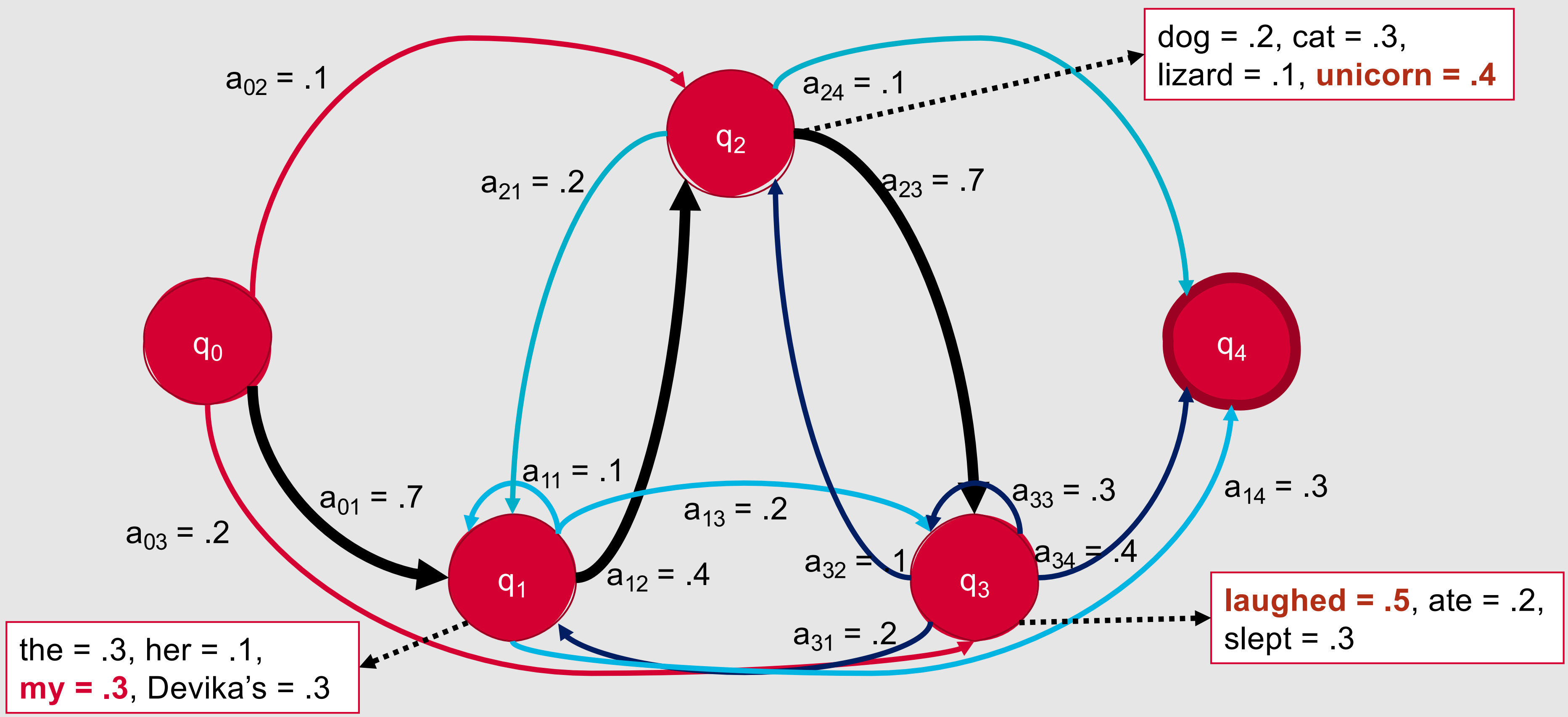
Sample Text Generation



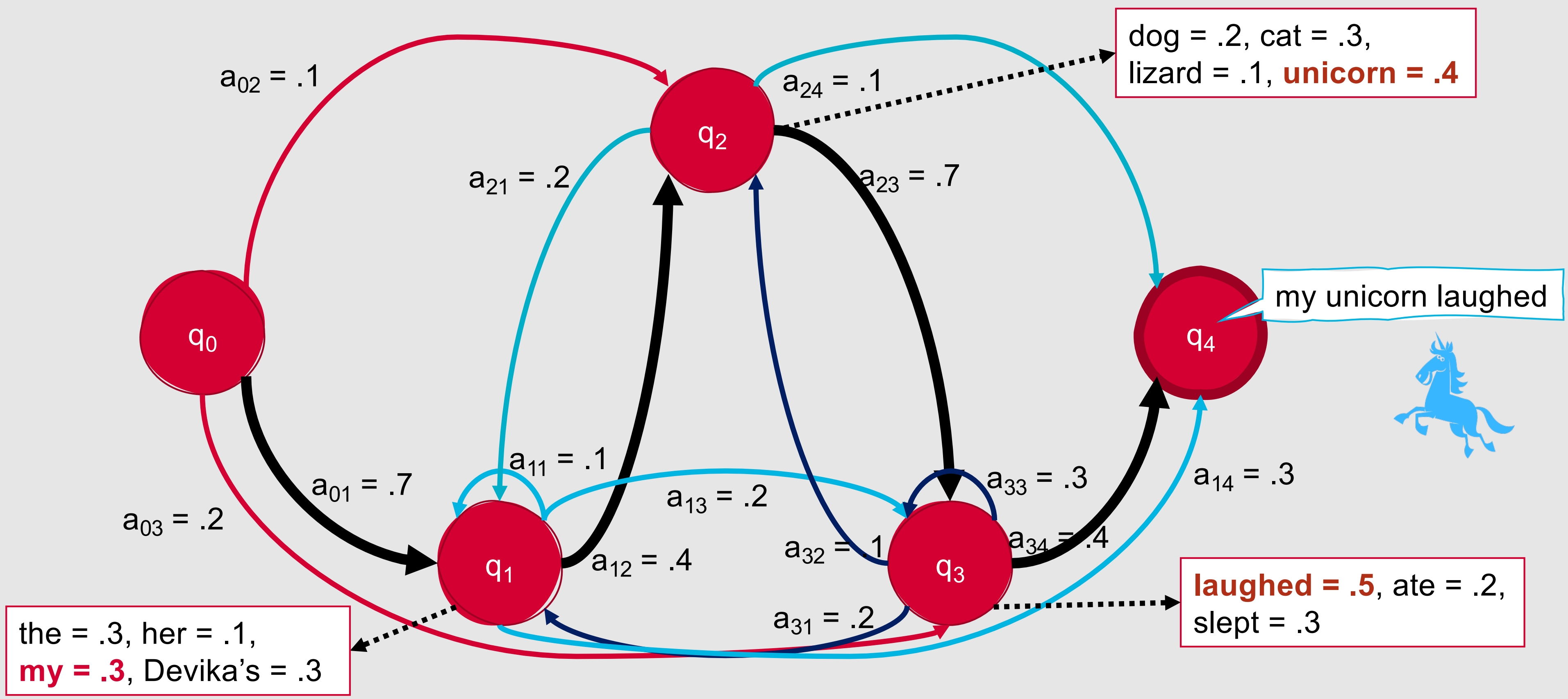
Sample Text Generation



Sample Text Generation



Sample Text Generation



Three Fundamental HMM Problems

- **Observation Likelihood:** How likely is a particular observation sequence to occur?
- **Decoding:** What is the best sequence of hidden states for an observed sequence?
  - What is the best sequence of labels for our test data?
- **Learning:** What are the transition probabilities and observation likelihoods that best fit the observation sequence and HMM states?
  - How do we empirically fit our training data?

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Observation Likelihood

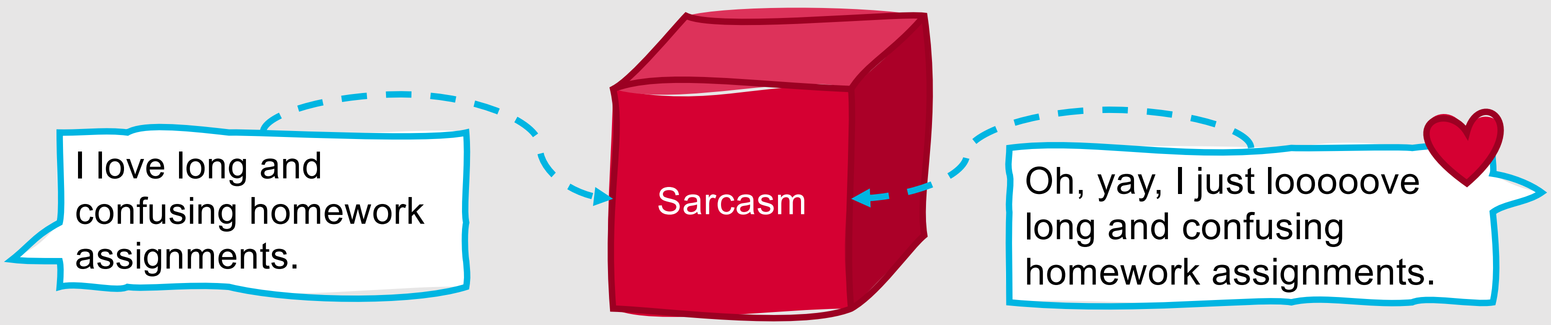
- Given a sequence of observations and an HMM, what is the probability that this sequence was generated by the model?
- Useful for two tasks:
  - Sequence classification
  - Selecting the most likely sequence

Sequence Classification

- Assuming an HMM is available for every possible class, what is the most likely class for a given observation sequence?
  - Which HMM is most likely to have generated the sequence?

Most Likely Sequence

- Of two or more possible sequences, which one was most likely generated by a given HMM?



How can we compute the observation likelihood?

- Naïve Solution:
  - Consider all possible state sequences, Q, of length T that the model, λ, could have traversed in generating the given observation sequence, O
  - Compute the probability of a given state sequence from A, and multiply it by the probability of generating the given observation sequence for that state sequence
    - $P(O,Q \mid \lambda) = P(O \mid Q, \lambda) * P(Q \mid \lambda)$
  - Repeat for all possible state sequences, and sum over all to get $P(O \mid \lambda)$
- But, this is computationally complex!
  - $O(TN^T)$

How can we compute the observation likelihood?

- Efficient Solution:
  - **Forward Algorithm:** Dynamic programming algorithm that computes the observation likelihood by summing over the probabilities of all possible hidden state paths that could generate the observation sequence.
  - Implicitly folds each of these paths into a single **forward trellis**
- Why does this work?
  - Markov assumption (the probability of being in any state at a given time *t* only relies on the probability of being in each possible state at time *t*-1)
- Works in $O(TN^2)$ time!

How does the forward algorithm work?

- Let $\alpha_i(j)$ be the probability of being in state *j* after seeing the first *t* observations, given your HMM λ
- $\alpha_i(j)$ is computed by summing over the probabilities of every path that could lead you to this cell
  - $\alpha_i(j) = P(o_1, o_2 \ldots o_t, q_t = j|\lambda) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t)$
  - $\alpha_{t-1}(i)$: The previous forward path probability from the previous time step
  - a_{ij}: The transition probability from previous state q_i to current state q_j
  - $b_j(o_t)$: The state observation likelihood of the observed item o_t given the current state *j*

Note the distinction between alpha (α) and a (a)!

Formal Algorithm

```
create a probability matrix forward[N+2,T]

for each state q in [1, …, N] do:
     forward[q,1] ← a_{0,q} * b_q(o_1)
for each time step t from 2 to T do:
     for each state q in [1, …, N] do:
          forward[q,t] ←
```

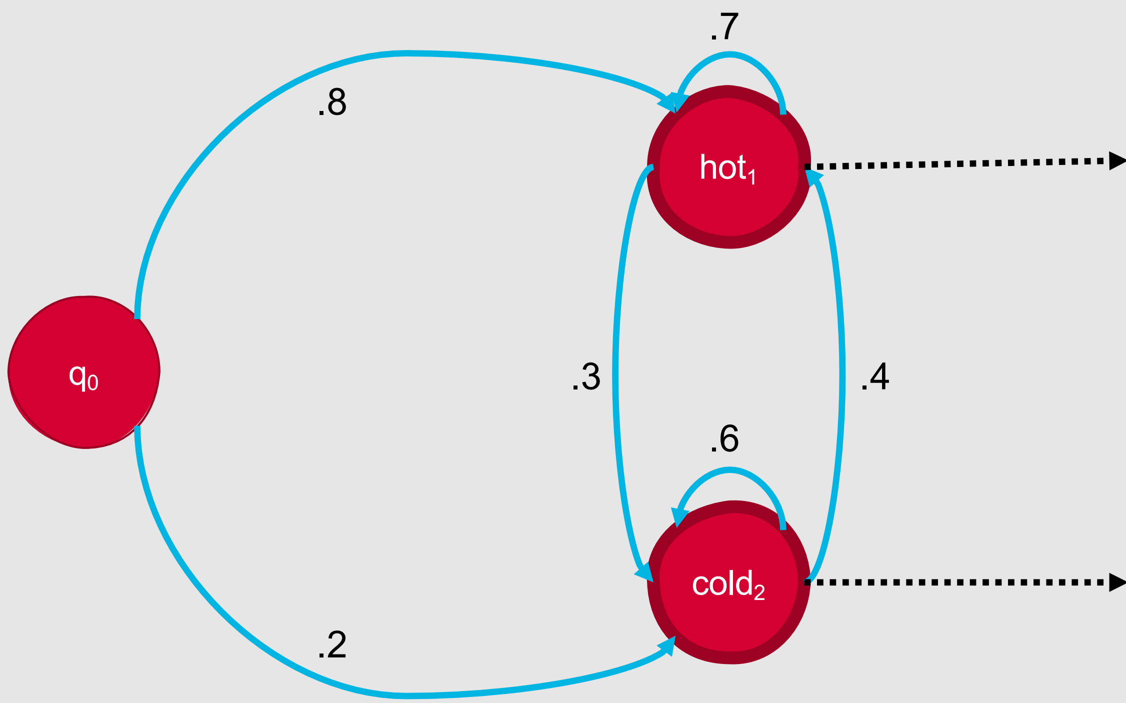
$\text{forward}[q,t] \leftarrow \sum_{q'=1}^{N} forward[q',t-1] * a_{q',q} * b_q(o_t)$

$\text{forwardprob} \leftarrow \sum_{q=1}^{N} forward[q,T]$

Sample Problem

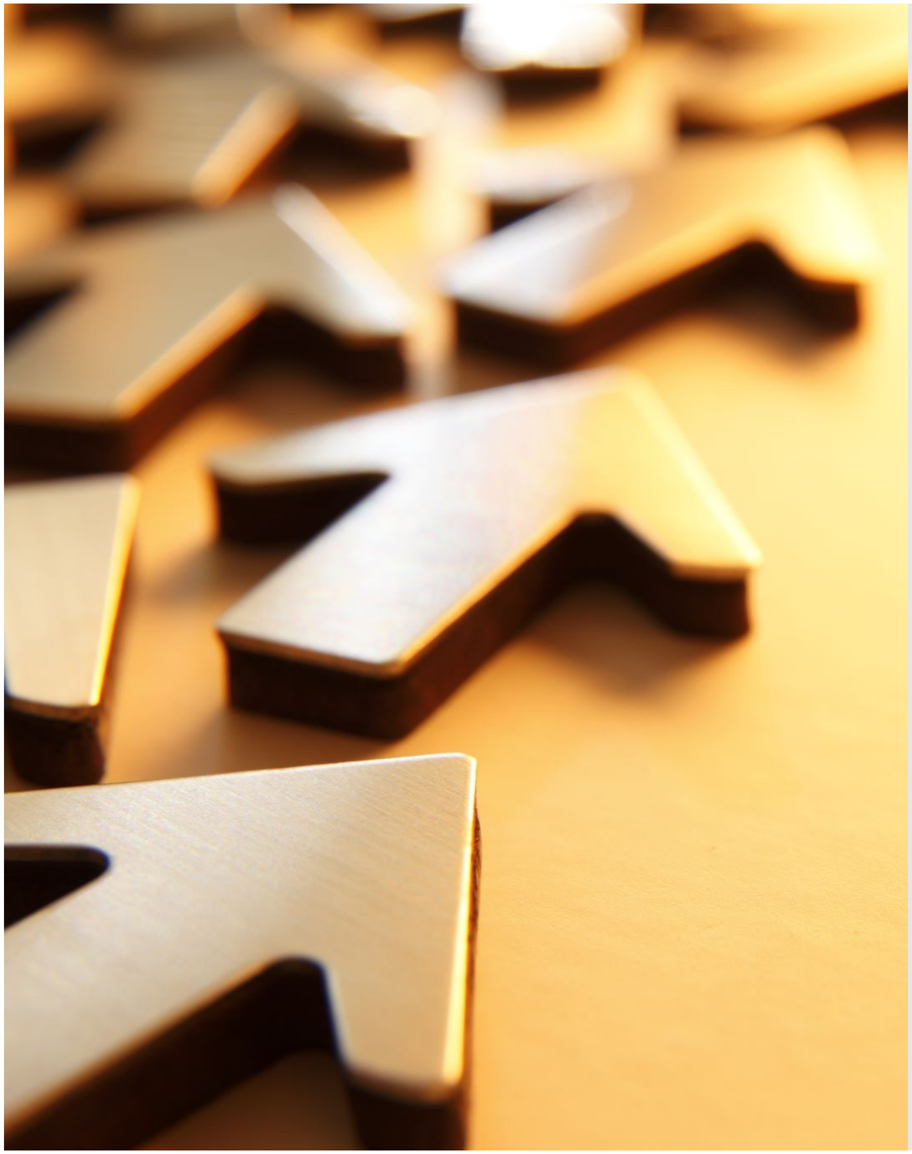
- You’re trying to solve a problem that relies on you knowing which days it was hot and cold in Chicago during the summer of 1924
- Unfortunately, you have no official records of the weather in Chicago for that summer, although you’re trying to model some key weather patterns from that year using an HMM
- You do have one promising lead: You find a detailed diary tracking how many ice cream cones the author of that diary ate on each day
- You decide to focus on a three-day sequence:
  - Day 1: 3 ice cream cones
  - Day 2: 1 ice cream cone
  - Day 3: 3 ice cream cones
- Your first task is to determine whether your current HMM does a good job at modeling this sequence

Your HMM



$$B_1 \quad \begin{bmatrix} P(1|hot_1) \\ P(2|hot_1) \\ P(3|hot_1) \end{bmatrix} = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}$$

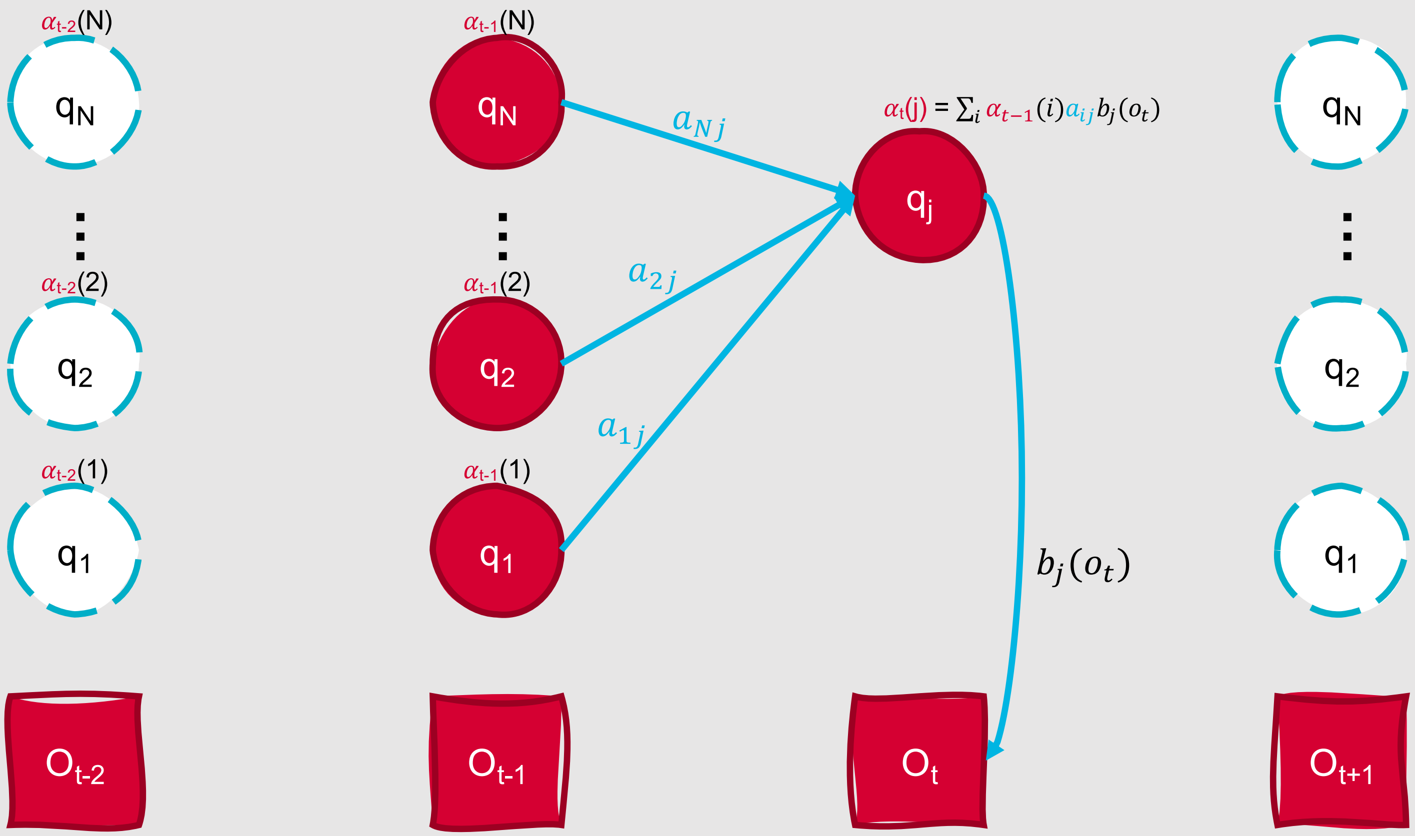
$$B_2 \quad \begin{bmatrix} P(1|cold_1) \\ P(2|cold_1) \\ P(3|cold_1) \end{bmatrix} = \begin{bmatrix} .5 \\ .4 \\ .1 \end{bmatrix}$$



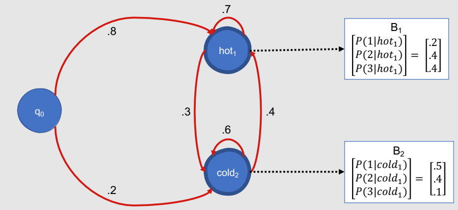
Forward Trellis

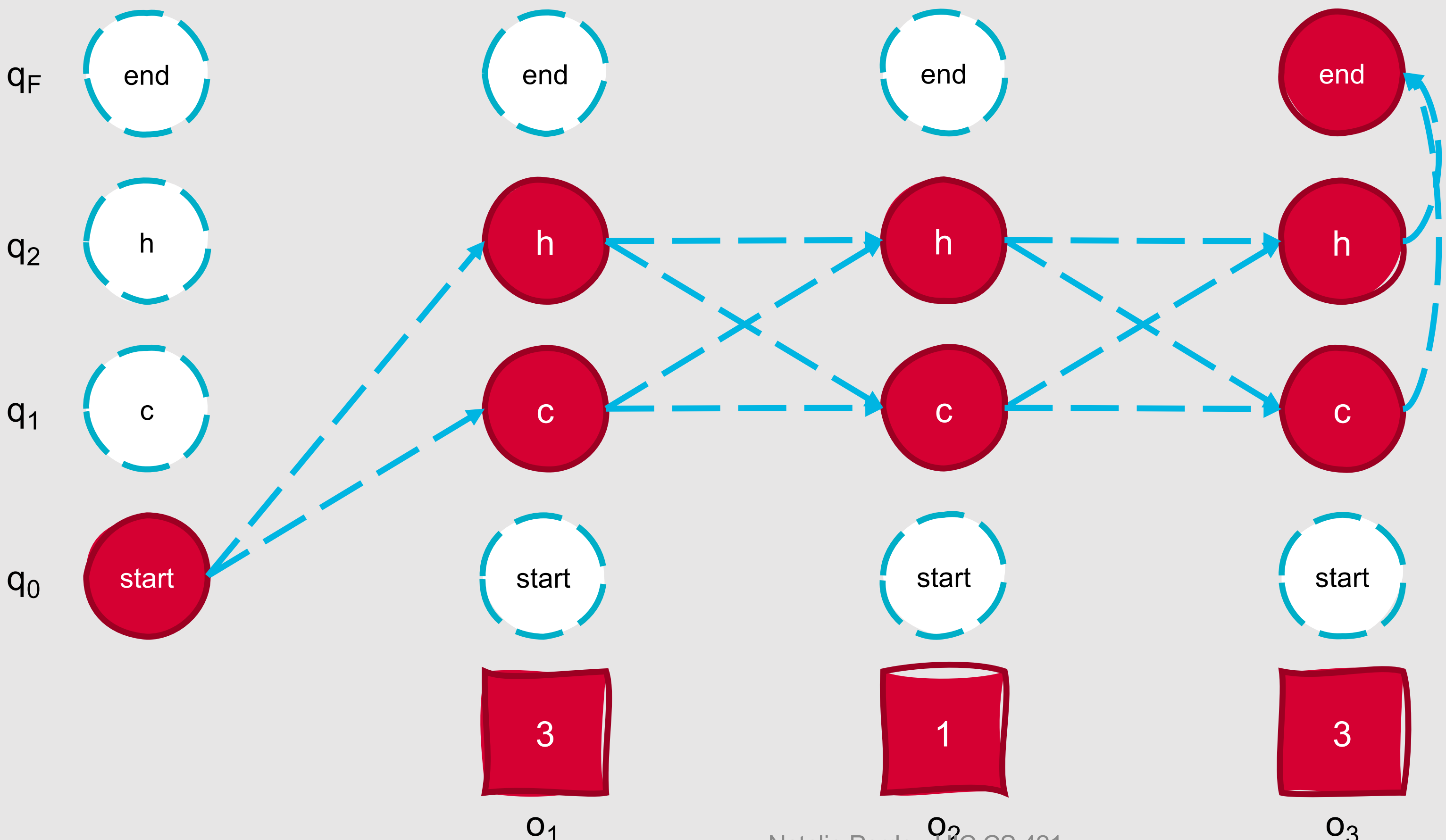
- Incorporates all the information you'll need to implement the forward algorithm
  - Observations
  - Transition probabilities
  - State observation likelihoods
  - Forward probabilities from earlier observations

Forward Step

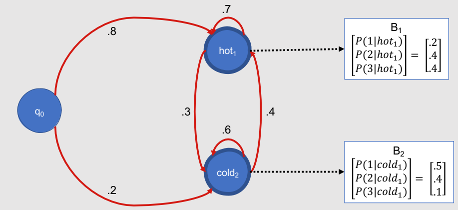


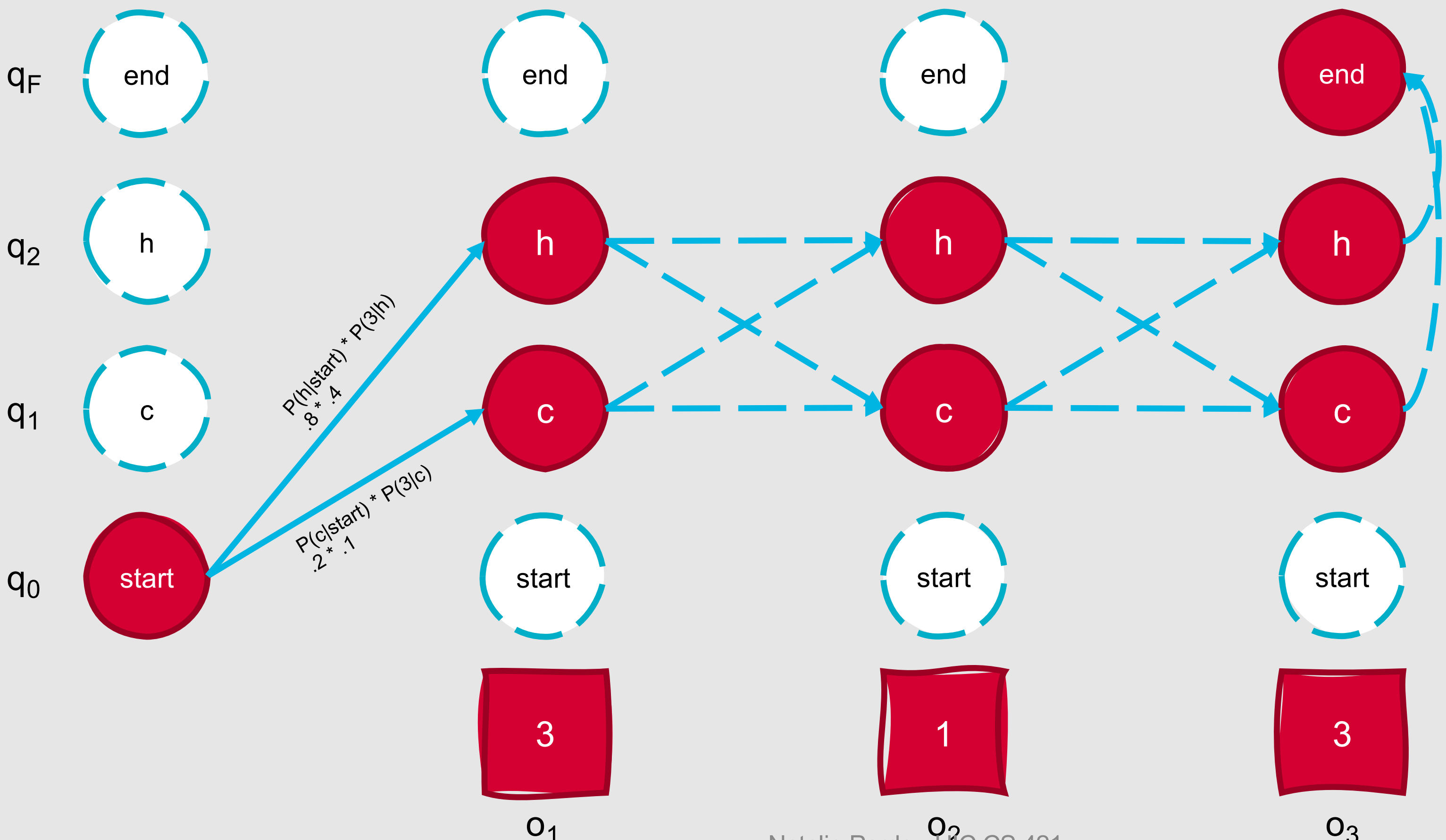
Forward Trellis



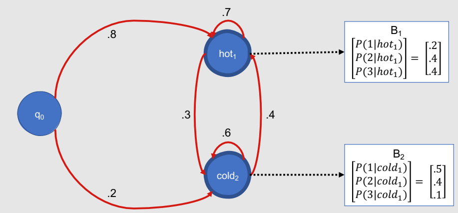


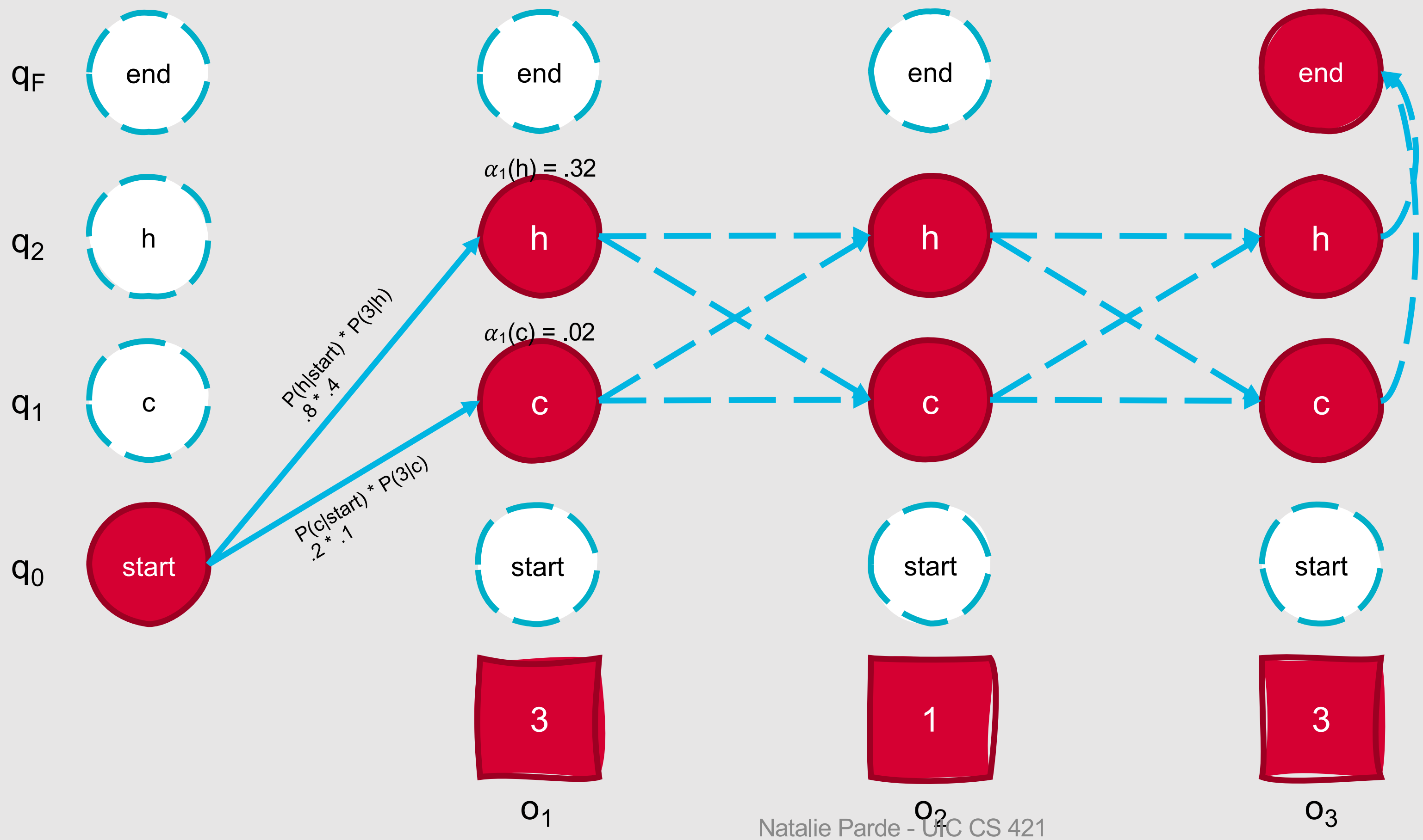
Forward Trellis





Forward Trellis





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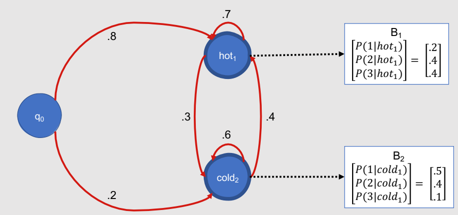

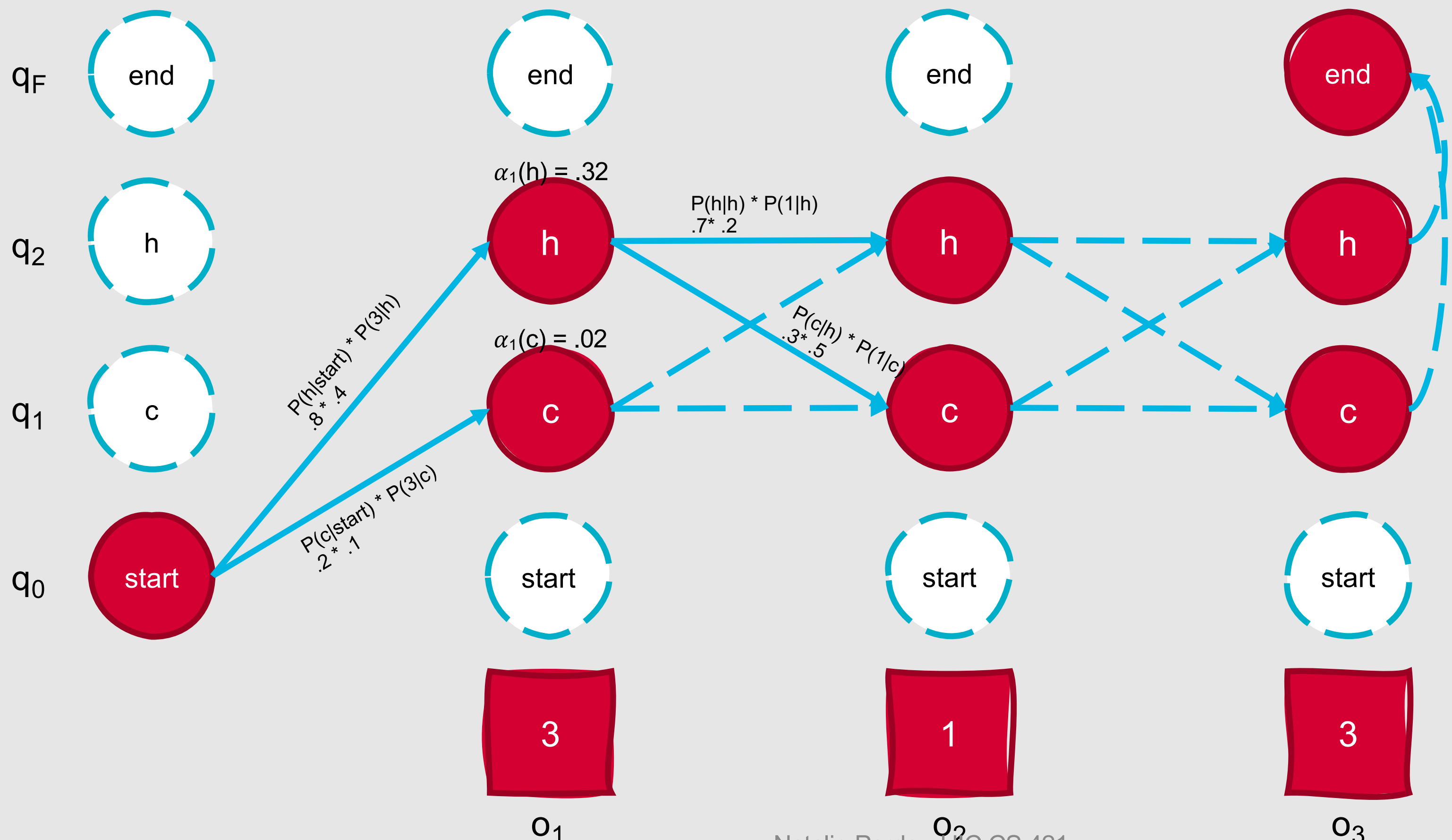

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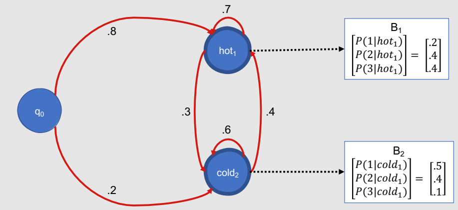

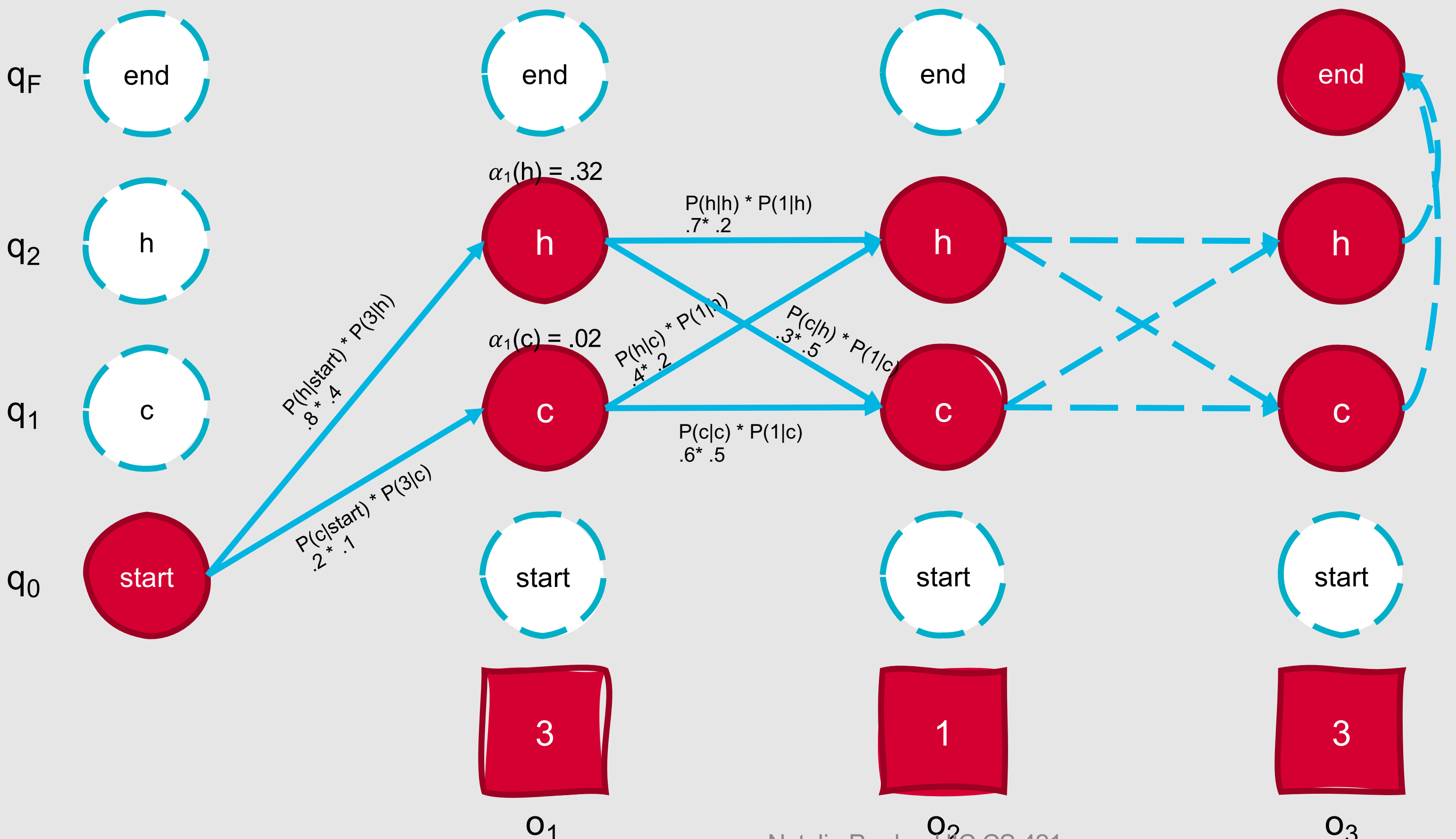

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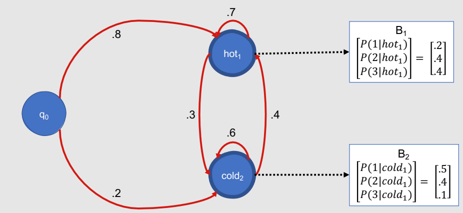

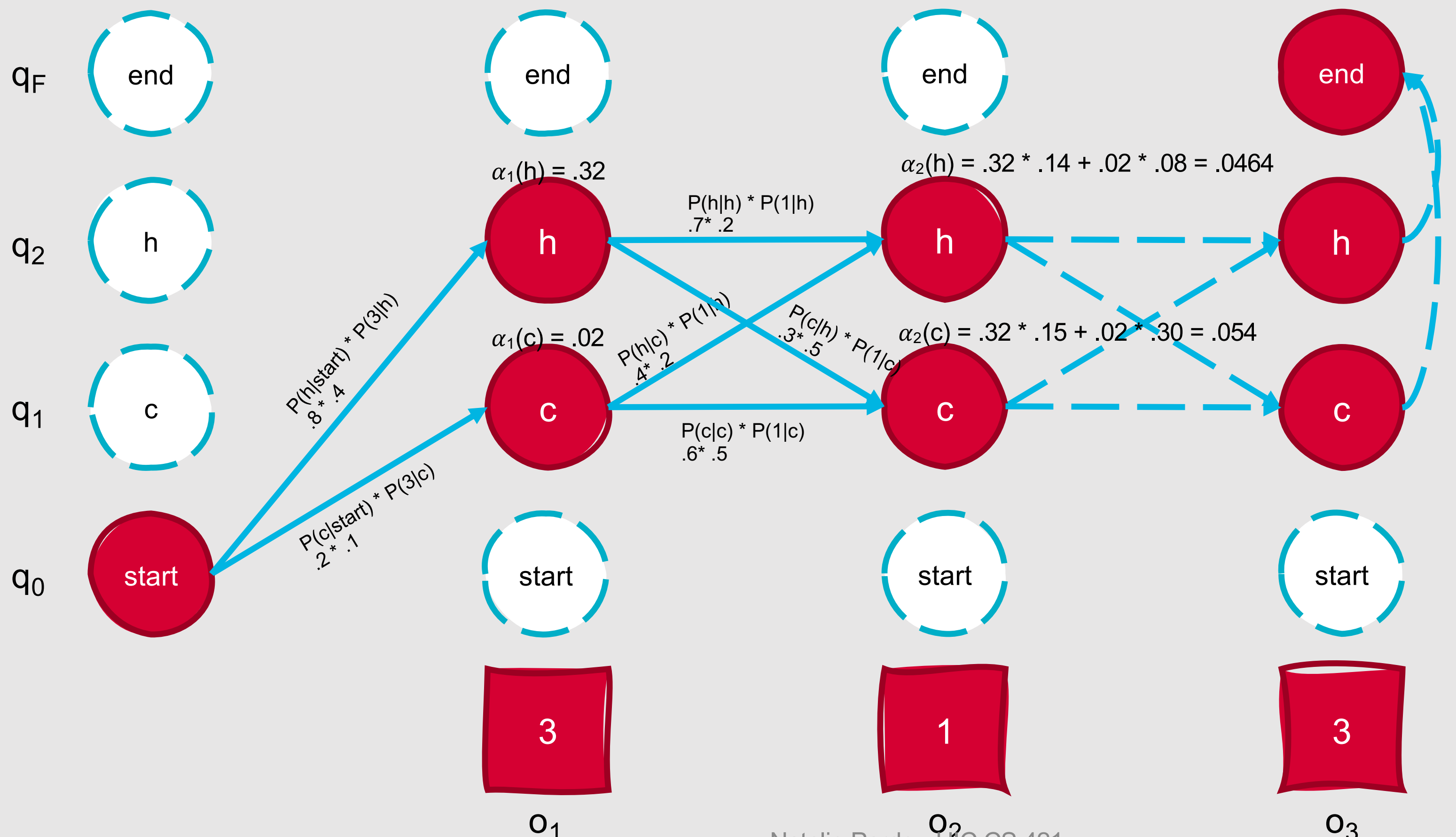

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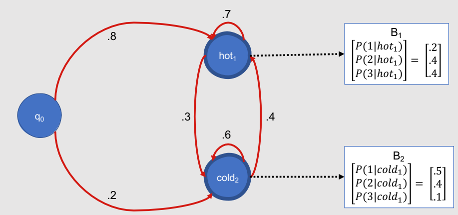

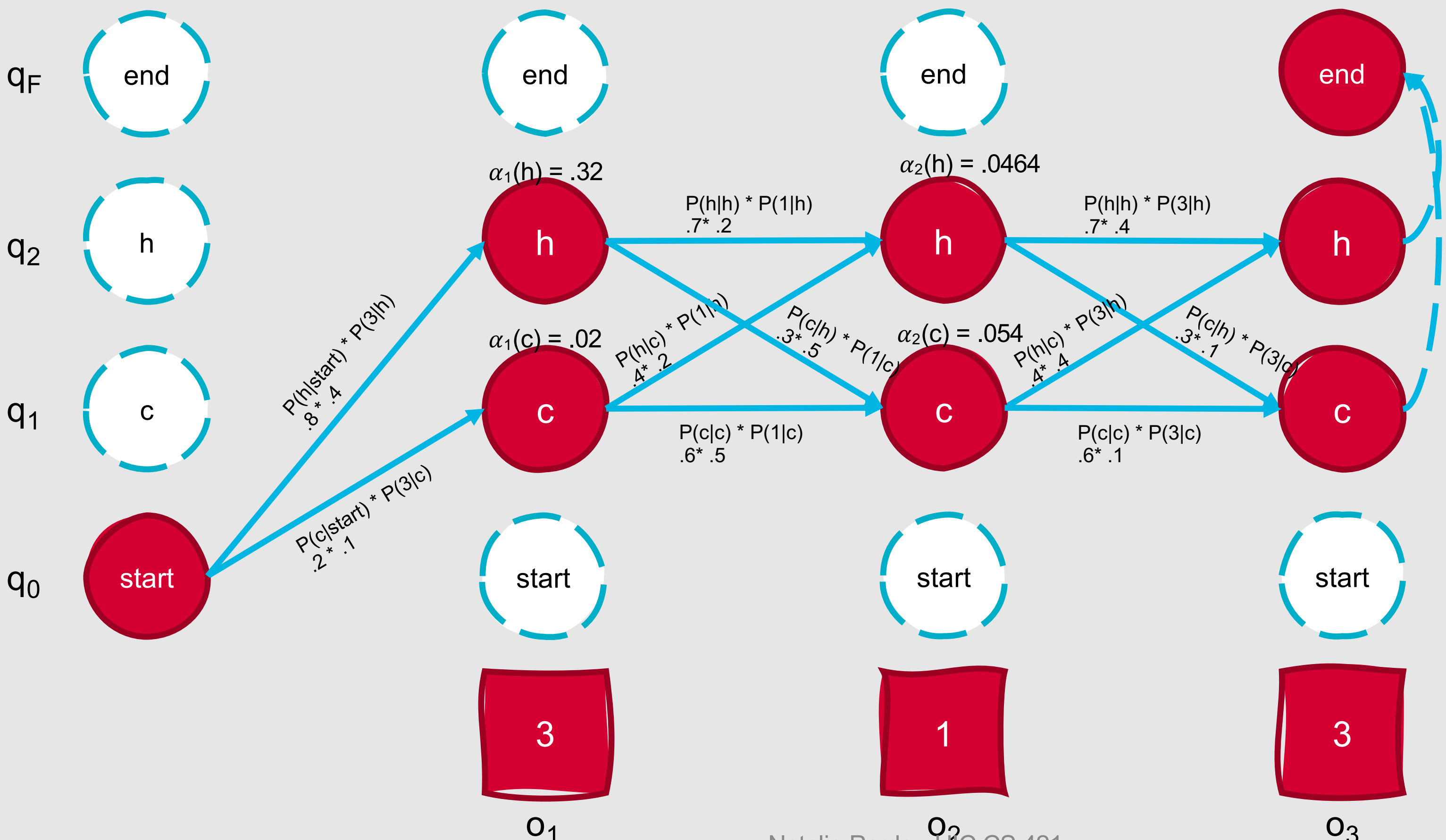

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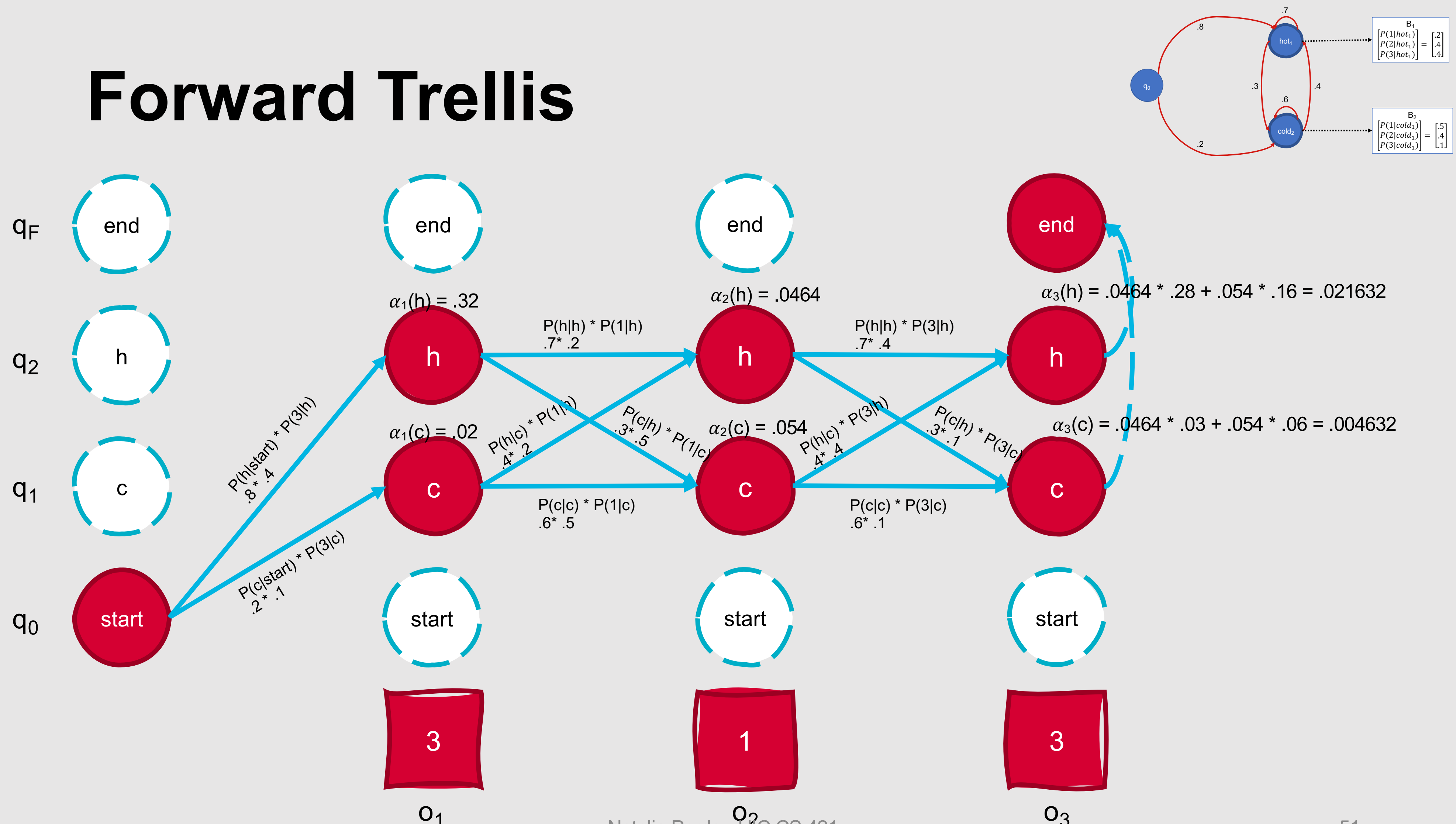

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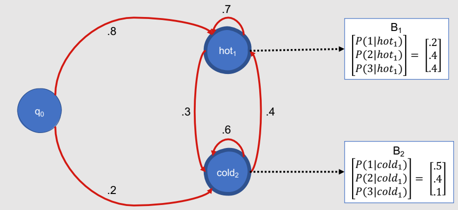
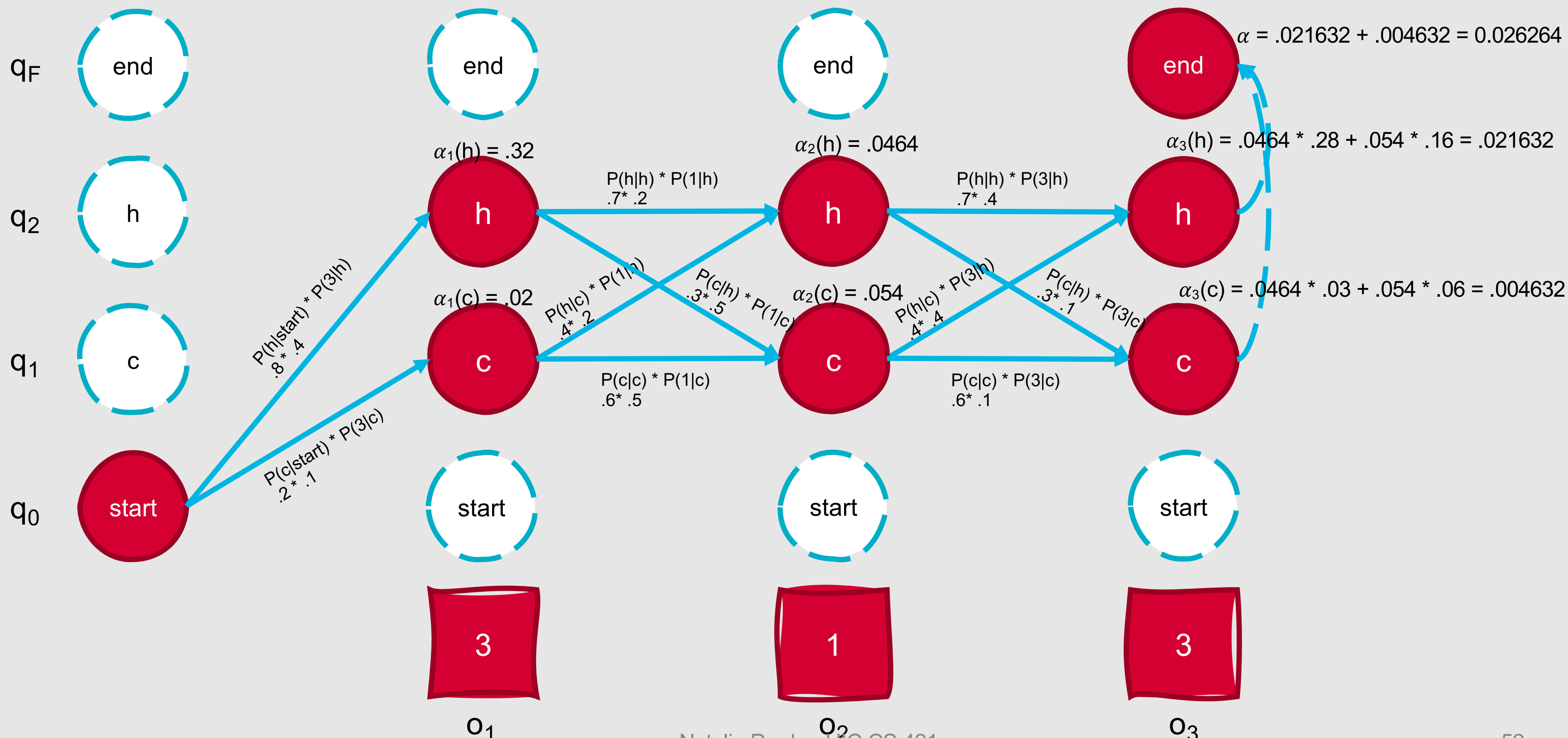




We've so far tackled one of the fundamental HMM tasks.

- What is the probability that a sequence of observations fits a given HMM?
  - Calculate using forward probabilities!
- However, there are still two remaining tasks to explore….

This Week's Topics

Tuesday

- N-gram language modeling
- Evaluating LMs
- Improving n-gram LMs

Thursday

- Hidden Markov Models
- Forward Algorithm
- Viterbi Algorithm
- Forward-Backward Algorithm

Decoding

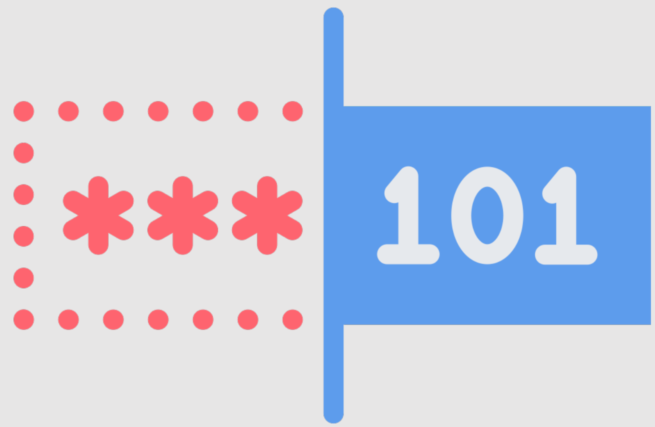
- Given an observation sequence and an HMM, what is the best hidden state sequence?
  - How do we choose a state sequence that is optimal in some sense (e.g., best explains the observations)?
- Very useful for sequence labeling!

Decoding

- Naïve Approach:
  - For each hidden state sequence Q, compute P(O|Q)
  - Pick the sequence with the highest probability
- However, this is computationally inefficient!
  - $O(N^T)$

How can we decode sequences more efficiently?

- **Viterbi Algorithm**
  - Another dynamic programming algorithm
  - Uses a similar trellis to the Forward algorithm
- Viterbi time complexity: $O(N^2T)$



Viterbi Intuition

- **Goal:** Compute the joint probability of the observation sequence together with the best state sequence
- So, **recursively compute the probability of the most likely subsequence of states** that accounts for the first *t* observations and ends in state q_j.
  - $v_t(j) = \max_{q_0, q_1, \ldots, q_{t-1}} P\left(q_0, q_1, \ldots, q_{t-1}, o_1, \ldots, o_t, q_t = q_j | \lambda\right)$
- Also **record backpointers** that subsequently allow you to backtrace the most probable state sequence
  - $bt_t(j)$ stores the state at time *t*-1 that maximizes the probability that the system was in state q_j at time *t*, given the observed sequence

Formal Algorithm

```
create a path probability matrix Viterbi[N+2,T]

for each state q in [1,…,N] do:
      Viterbi[q,1] ← a_{0,q} * b_q(o_1)
      backpointer[q,1] ← 0
for each time step t in [2,…,T] do:
      for each state q in [1,…,N] do:
```

$$viterbi[q,t] \leftarrow \max_{q' \in [1,\ldots,N]} viterbi[q',t-1] * a_{q',q} * b_q(o_t)$$

$$backpointer[q,t] \leftarrow \operatorname*{argmax}_{q' \in [1,\ldots,N]} viterbi[q',t-1] * a_{q',q} * b_q(o_t)$$

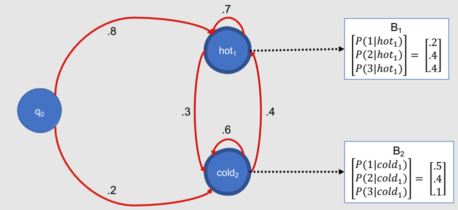
bestpathprob $\leftarrow \max_{q' \in [1,\ldots,N]} viterbi[q',T]$

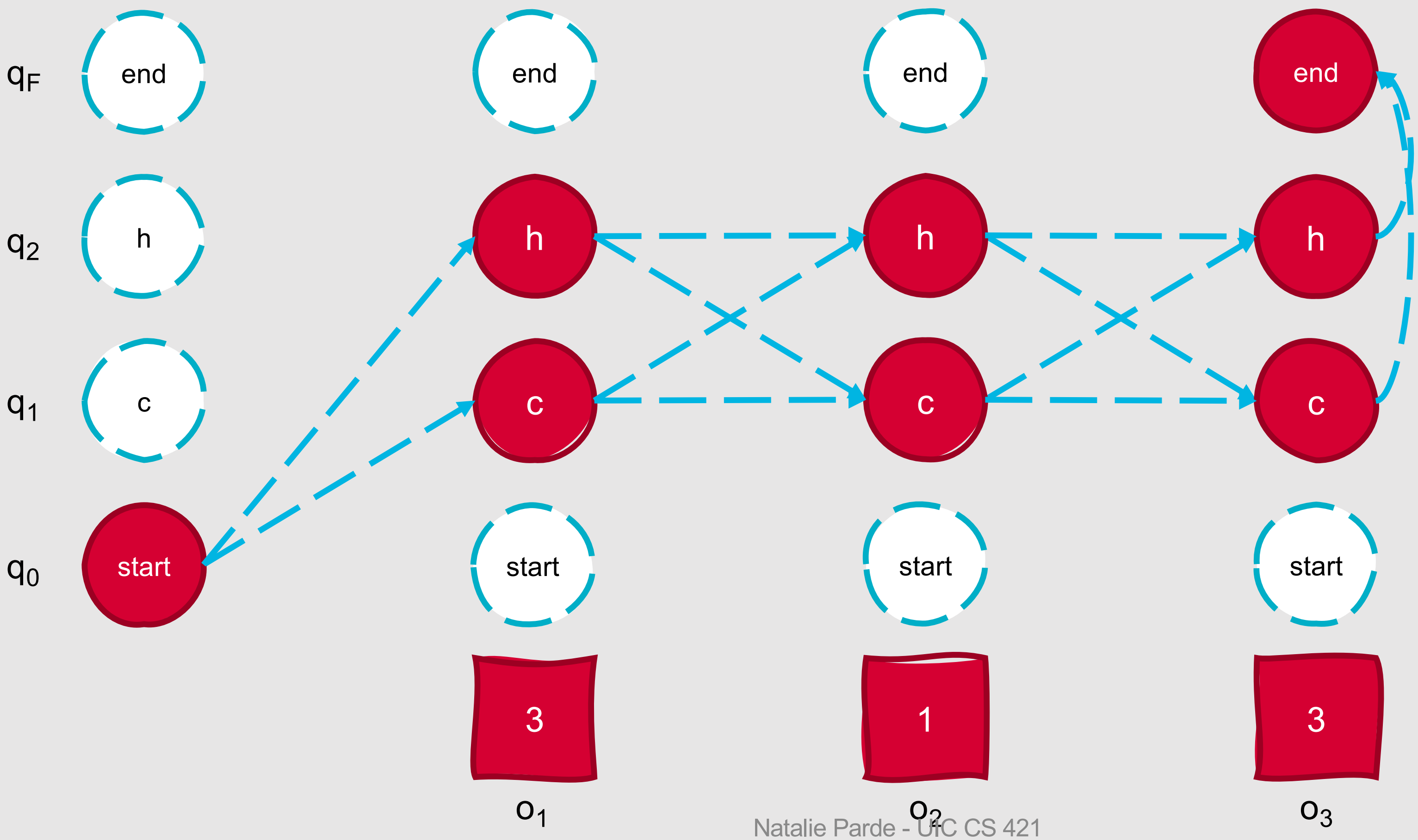
bestpathpointer $\leftarrow \operatorname*{argmax}_{q' \in [1,\ldots,N]} viterbi[q',T]$

Seem familiar?

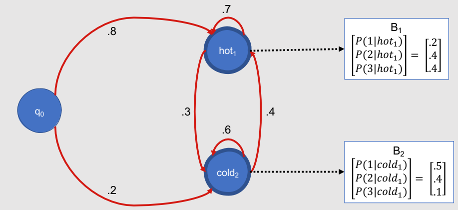
- Viterbi is basically the forward algorithm + backpointers!
- Instead of summing across prior forward probabilities, we use a max function

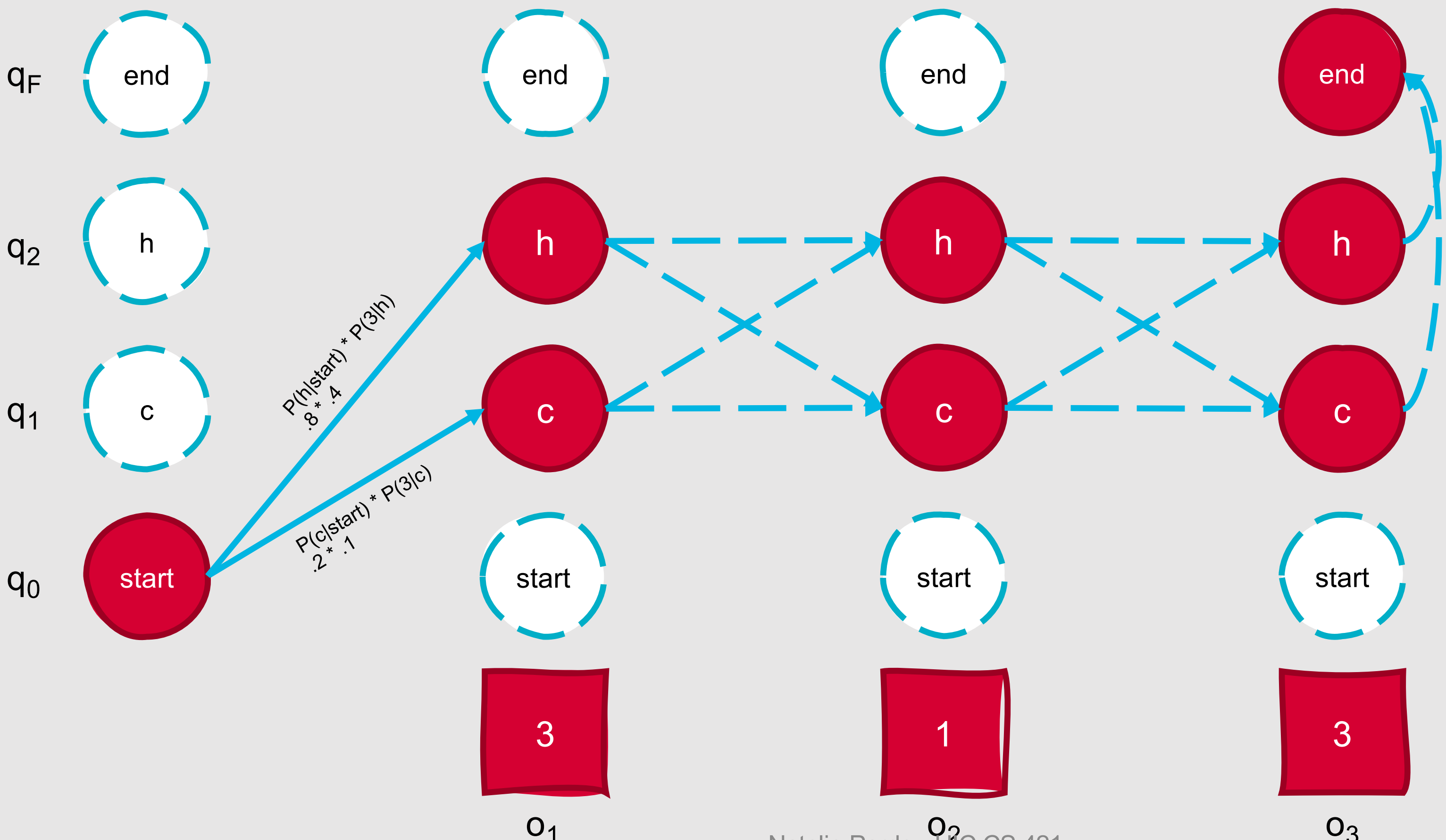
Viterbi Trellis



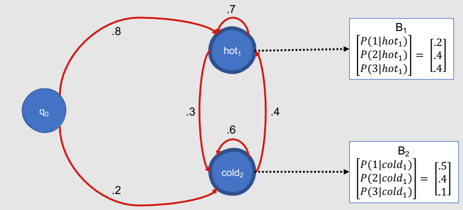


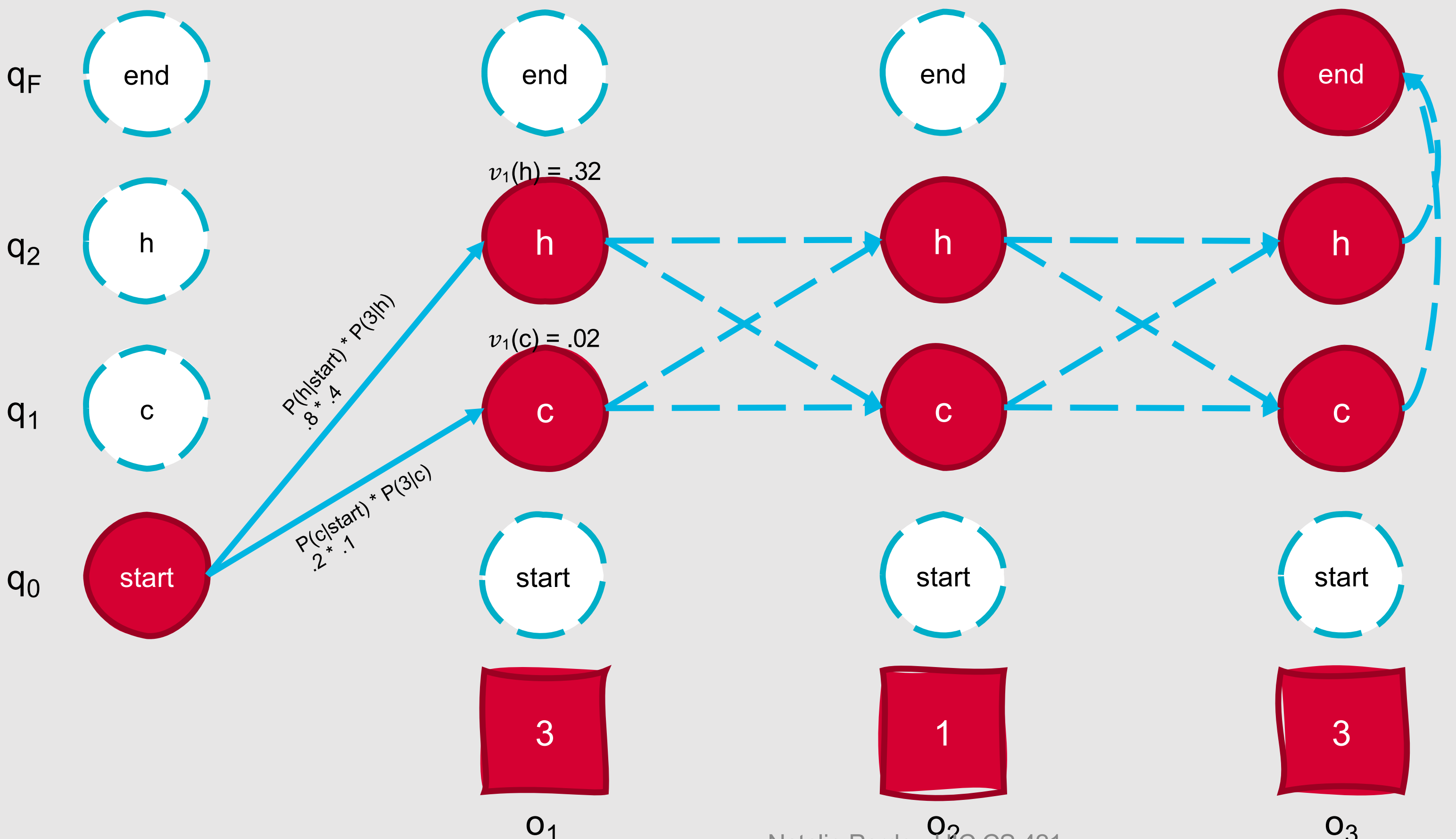
Viterbi Trellis



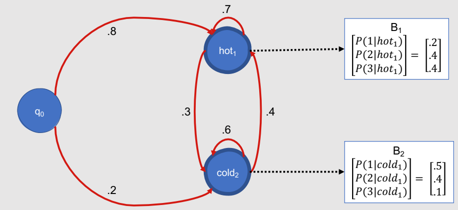


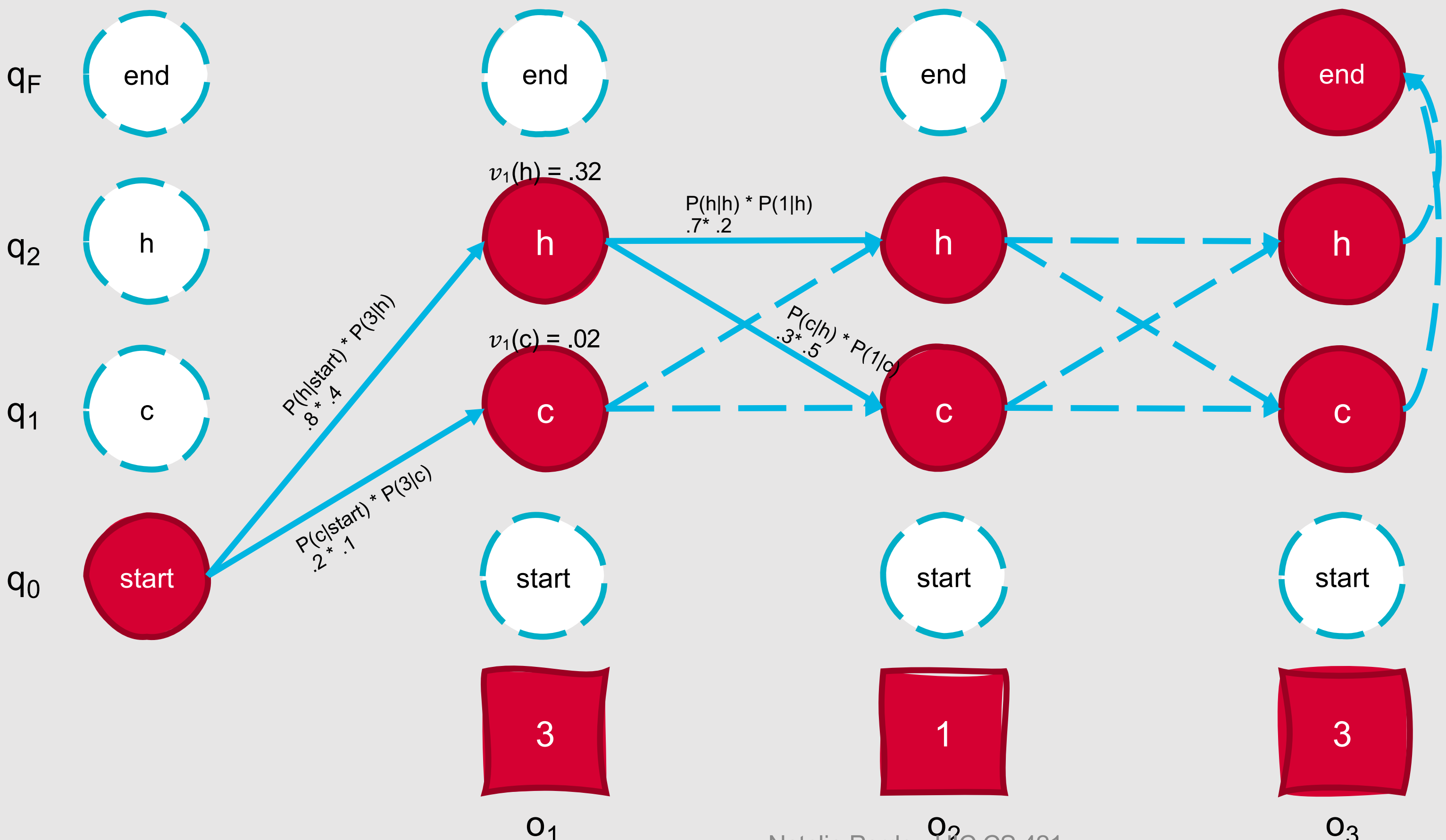
Viterbi Trellis



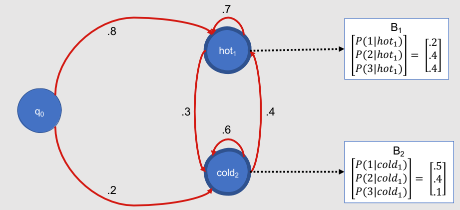


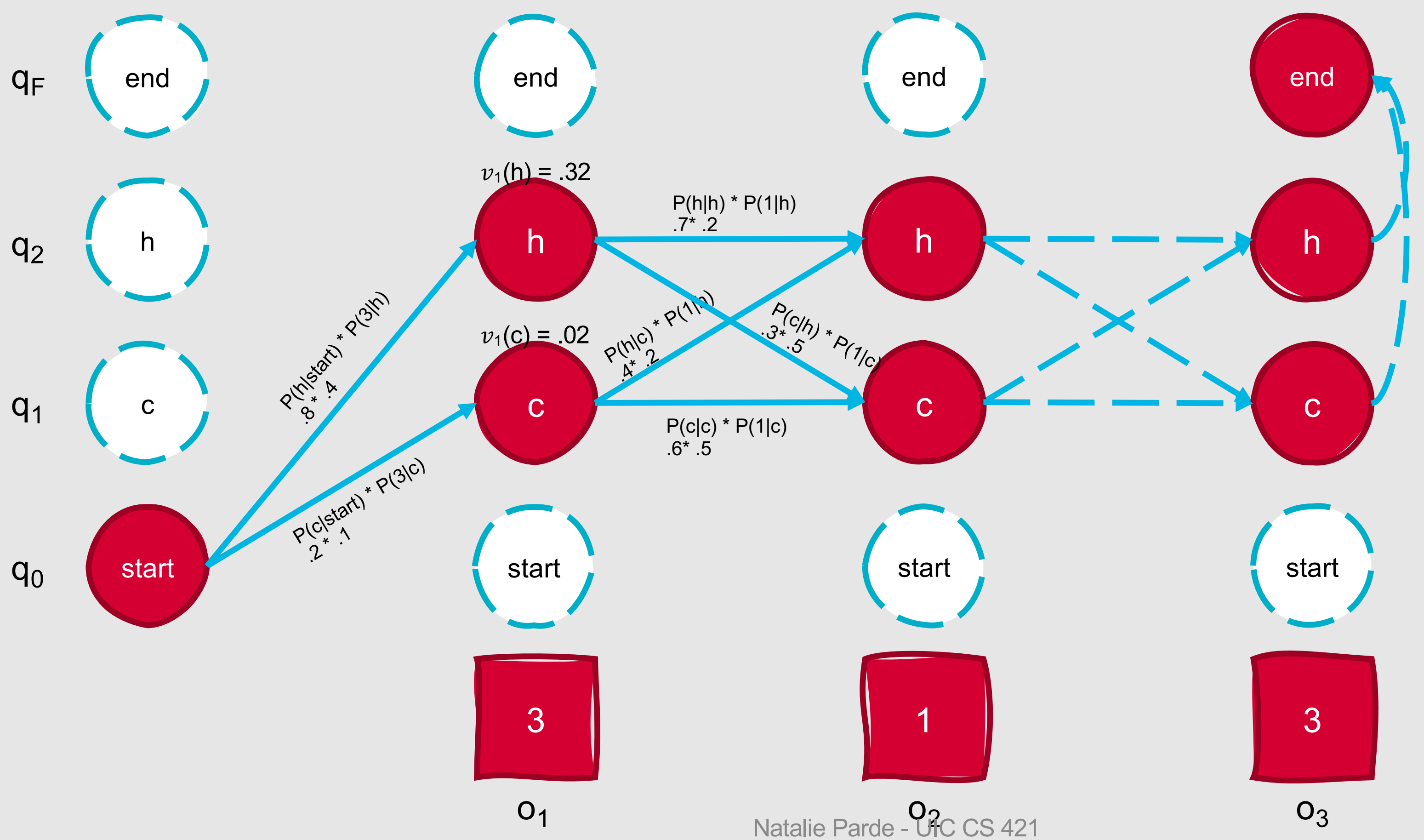
Viterbi Trellis



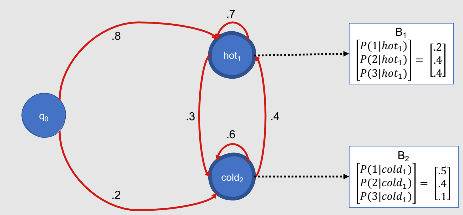


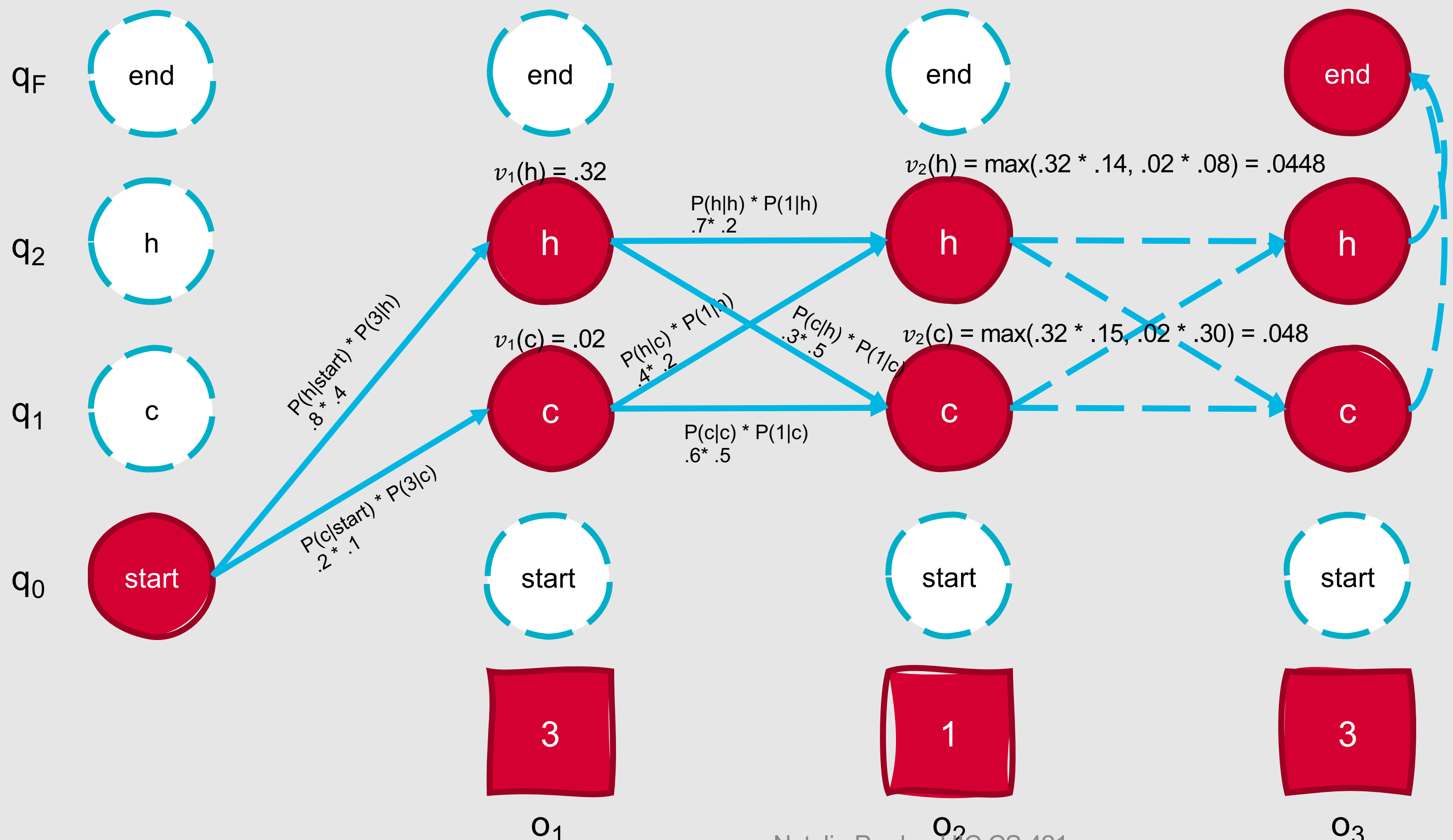
Viterbi Trellis



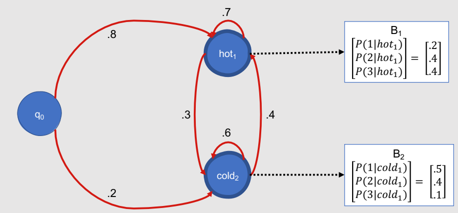


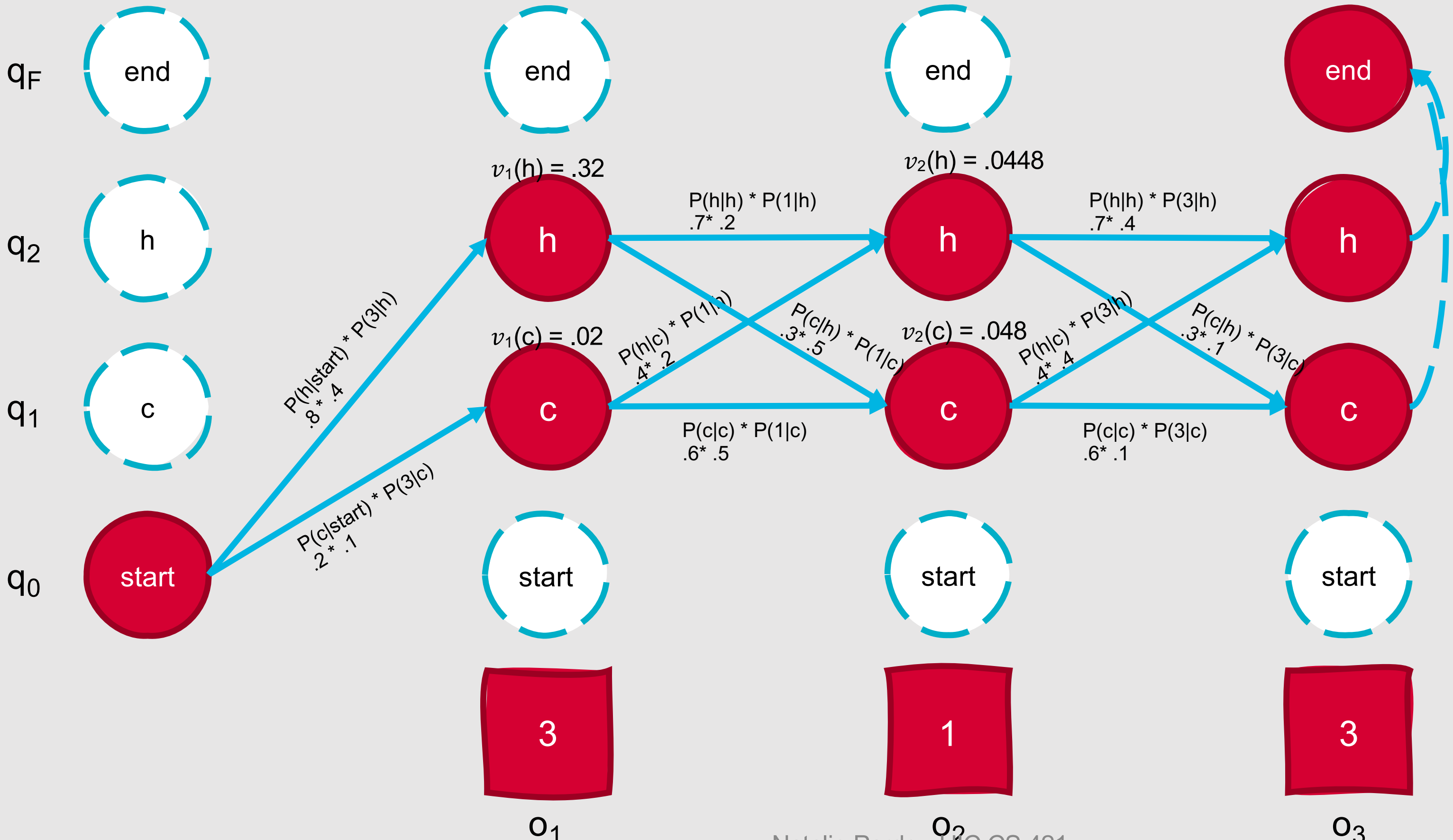
Viterbi Trellis



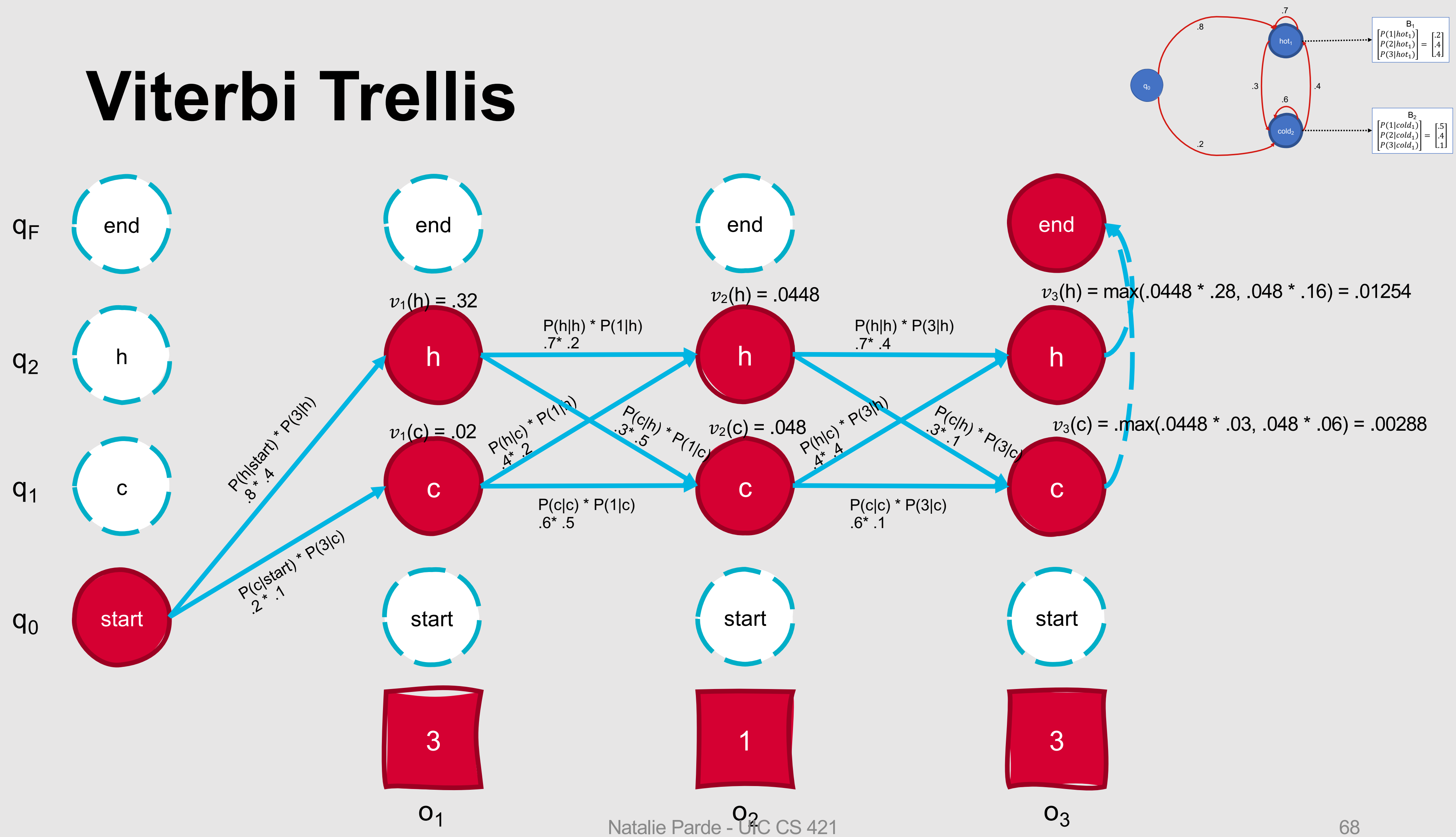


Viterbi Trellis

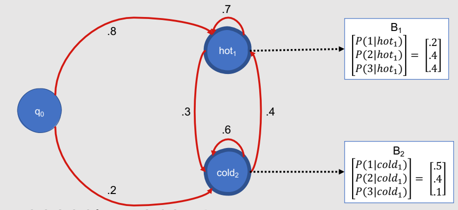


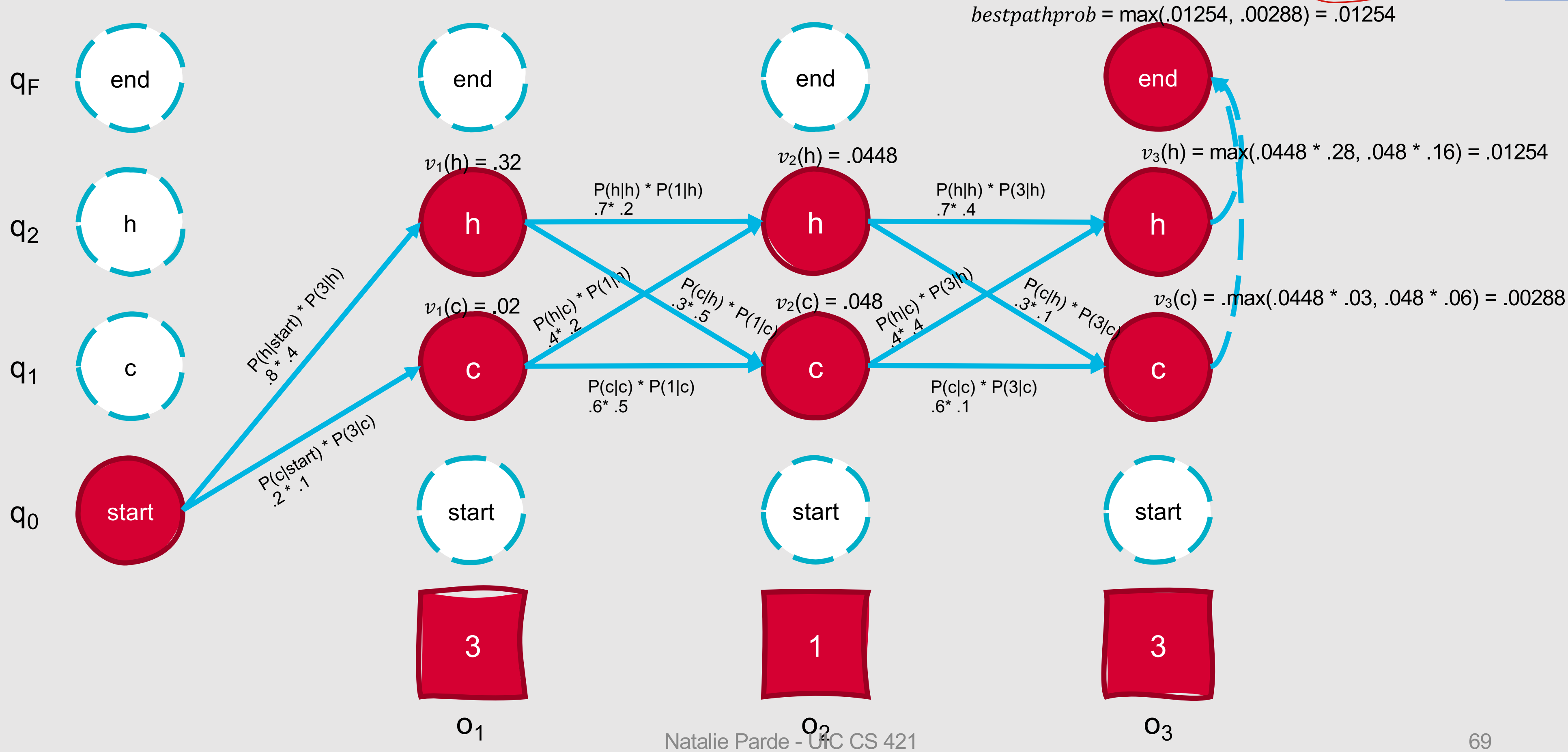


Viterbi Trellis

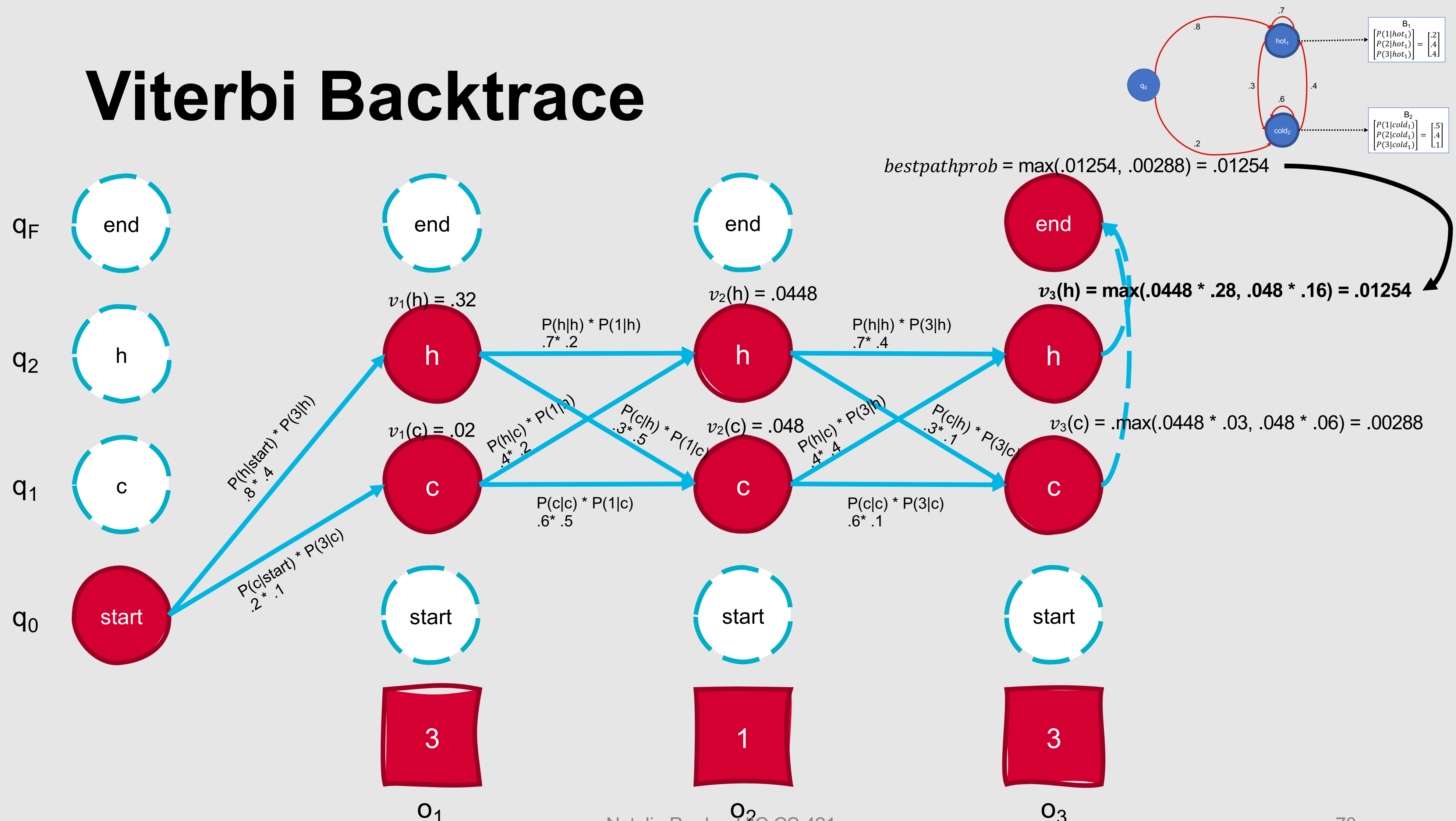


Viterbi Trellis

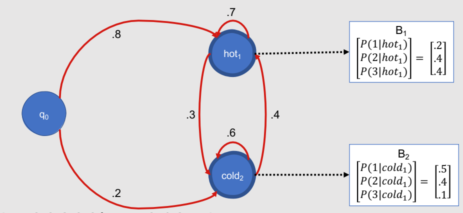


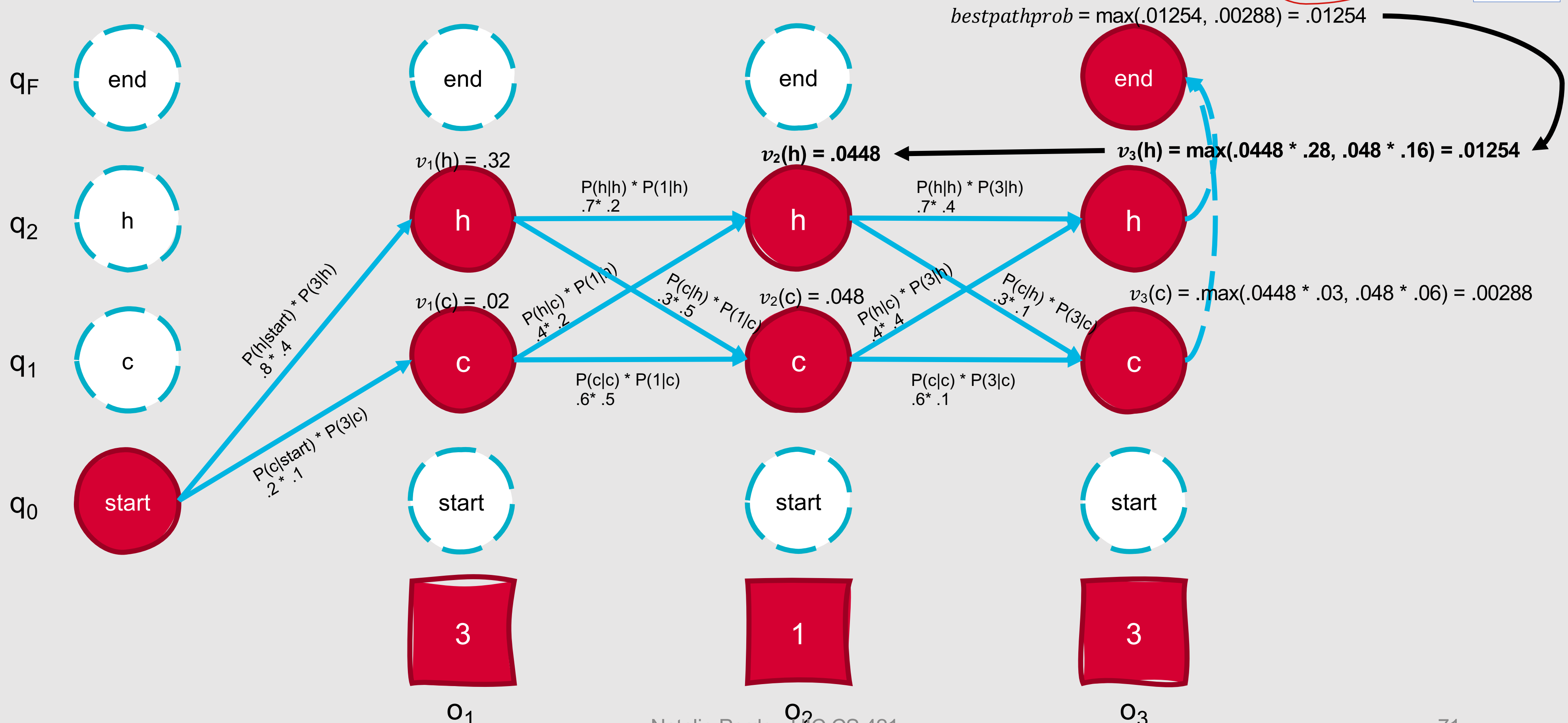


Viterbi Backtrace

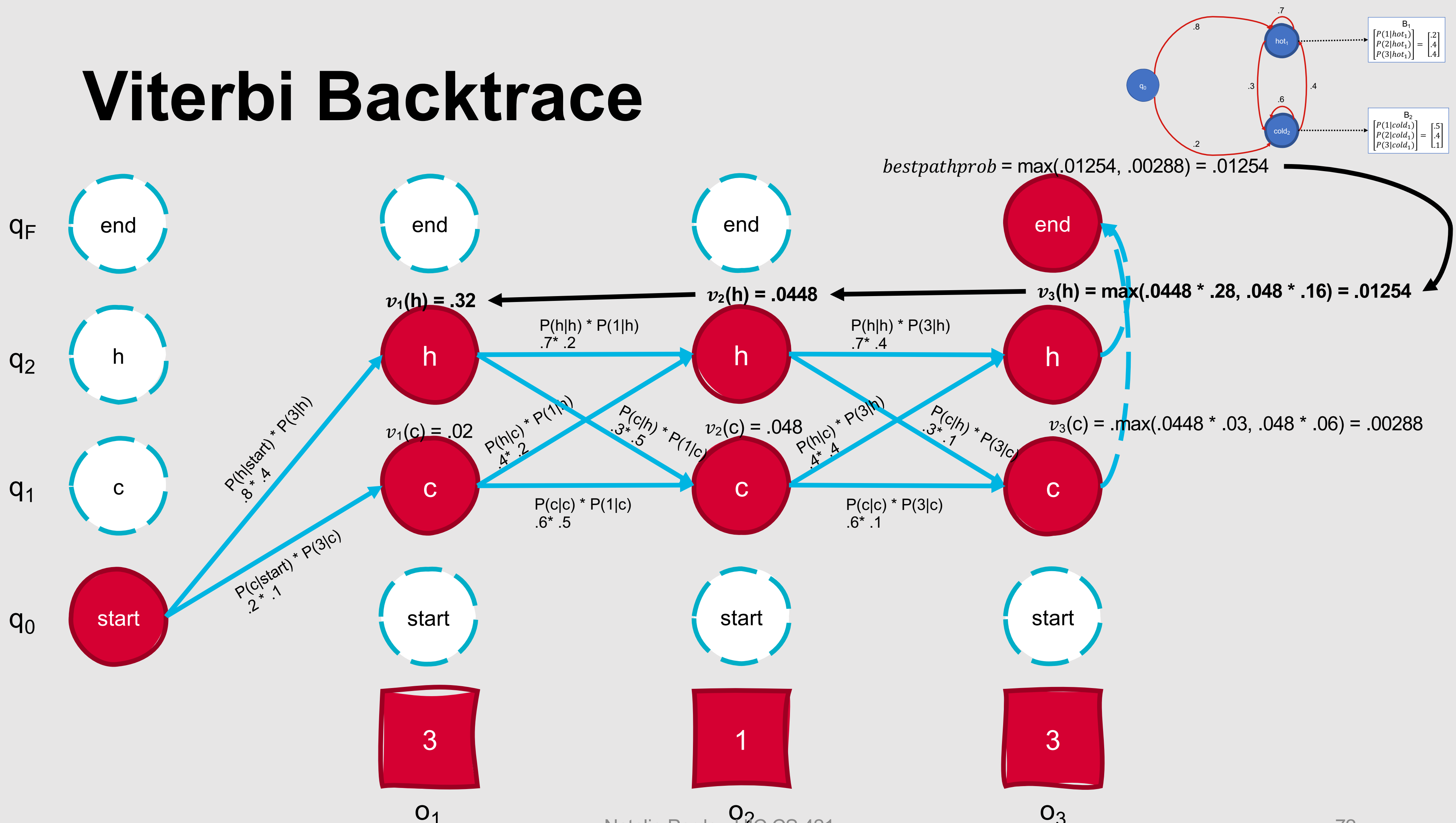


Viterbi Backtrace

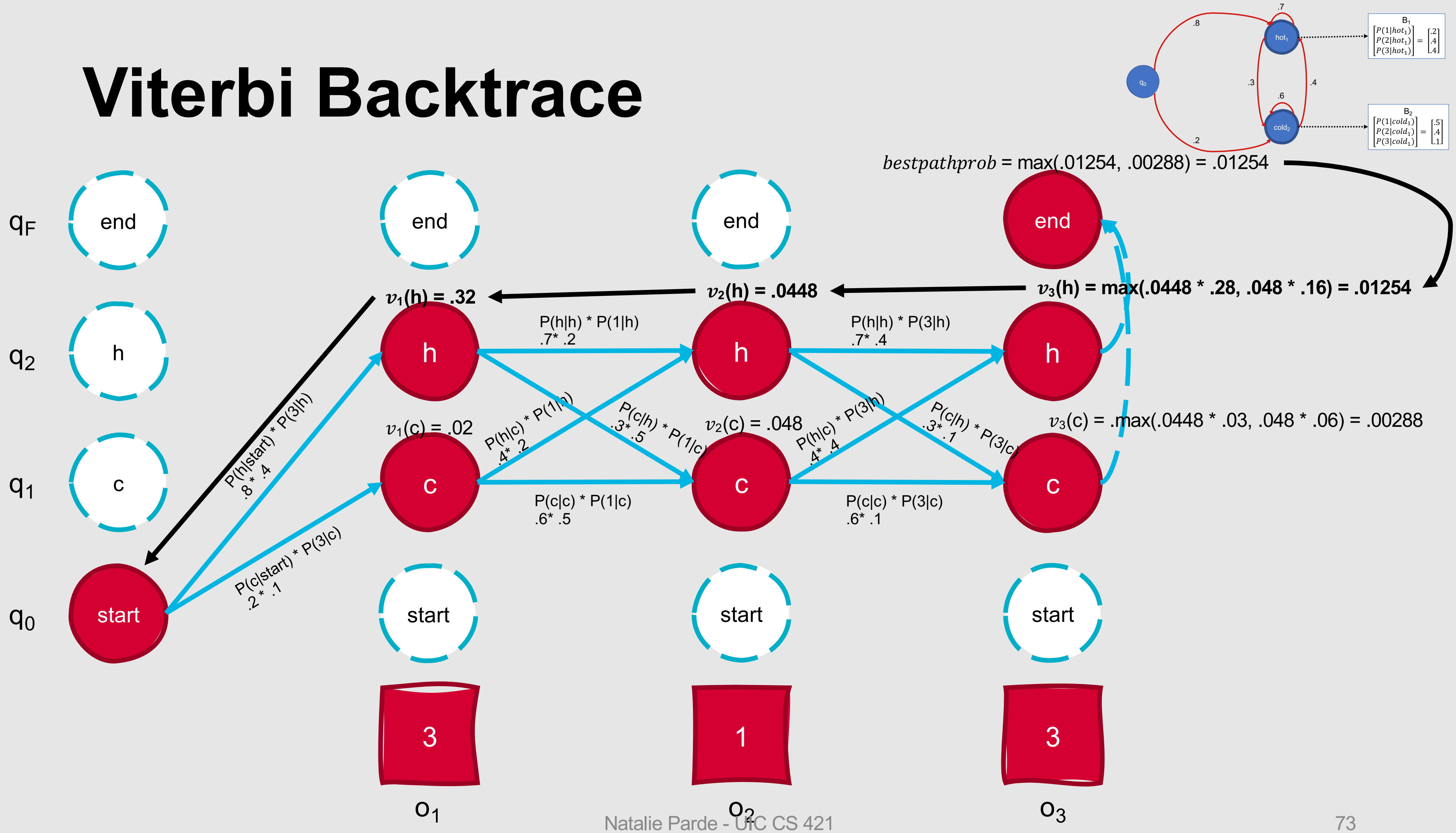




Viterbi Backtrace



Viterbi Backtrace



The Viterbi algorithm is used in many domains, even beyond text processing!

- **Speech recognition**
  - Given an input acoustic signal, find the most likely sequence of words or phonemes
- **Digital error correction**
  - Given a received, potentially noisy signal, determine the most likely transmitted message
- **Computer vision**
  - Given noisy measurements in video sequences, estimate the most likely trajectory of an object over time
- **Economics**
  - Given historical data, predict financial market states at certain timepoints

This Week's Topics

Tuesday

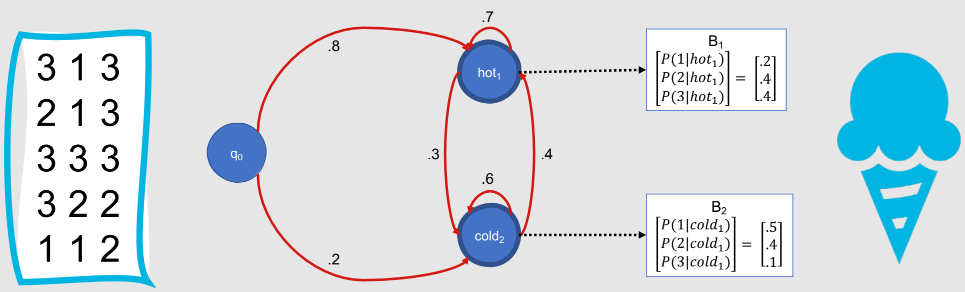
- N-gram language modeling
- Evaluating LMs
- Improving n-gram LMs

Thursday

- Hidden Markov Models
- Forward Algorithm
- Viterbi Algorithm
- Forward-Backward Algorithm

Finally …how do we train HMMs?

- If we have a set of observations, can we learn the parameters (transition probabilities and observation likelihoods) directly?



Forward-Backward Algorithm

- Special case of expectation-maximization (EM) algorithm
- Input:
  - Unlabeled sequence of observations, O
  - Vocabulary of hidden states, Q
- Output: Transition probabilities and observation likelihoods

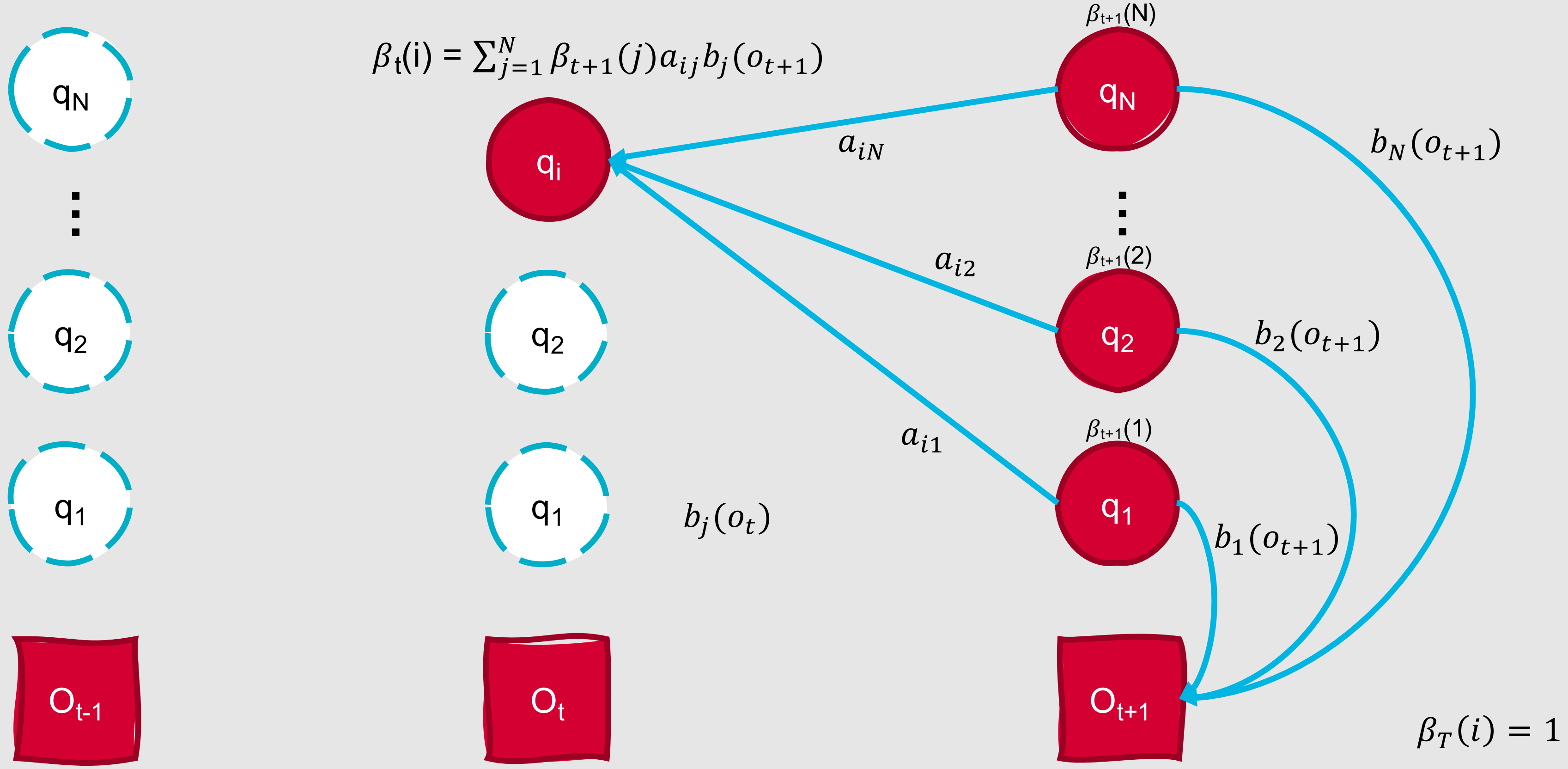
How does the algorithm compute these outputs?

- **Iteratively estimate the counts for transitions from one state to another**
  - Start with base estimates for a_{ij} and b_j, and iteratively improve those estimates
- **Get estimated probabilities** by:
  - Computing the forward probability for an observation
  - Dividing that probability mass among all the different paths that contributed to this forward probability (**backward probability**)

Backward Algorithm

- We define the backward probability as follows:
  - $\beta_t(i) = P(o_{t+1}, o_{t+2}, \ldots, o_T | q_t = i, \lambda)$
  - Probability of generating partial observations from time *t*+1 until the end of the sequence, given that the HMM λ is in state *i* at time *t*
- Also computed using a trellis, but moves backwards instead

Backward Step



For the expectation step of the forward-backward algorithm, we re-estimate transition probabilities and observation likelihoods.

- We re-estimate transition probabilities, a_{ij}, as follows:
  - Let $\zeta_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{\sum_{j=1}^{N} \alpha_t(i)\beta_t(j)}$
  - Then, $\widehat{a_{ij}} = \frac{\text{expected \# transitions from state } i \text{ to state } j}{\text{expected \# transitions from state } i} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1}\sum_{k=1}^{N} \xi_t(i,k)}$
  - Check out the course textbook (Appendix A) for an in-depth discussion of how the numerator and denominator above are derived!

Re-Estimating Observation Likelihood

- We re-estimate b_j as follows:
  - Let $\gamma_t(j) = \frac{\alpha_t(j)\beta_t(j)}{\alpha_T(q_F)}$
  - Then, $\hat{b}_j(v_k) = \frac{\text{expected \# of times in state } j \text{ and observing symbol } v_k}{\text{expected \# of times in state } j} = \frac{\sum_{t=1 \text{ s.t. } o_t=v_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$

Putting it all together, we have the forward-backward algorithm!

```
initialize A and B
iterate until convergence:

    # Expectation Step
    compute γ_t(j) for all t and j
    compute ζ_t(i,j) for all t, i, and j

    # Maximization Step
```

$$\widehat{a_{ij}} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \sum_{k=1}^{N} \xi_t(i,k)}$$

$$\widehat{b}_j(v_k) = \frac{\sum_{t=1 \text{ s.t. } o_t=v_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$$ for the symbol v_k in the vocabulary

Summary: Hidden Markov Models

- **HMMs** are probabilistic generative models for sequences
- They make predictions based on underlying hidden states
- Three fundamental HMM problems include:
  - Computing the likelihood of a sequence of observations
  - Determining the best sequence of hidden states for an observed sequence
  - Learning HMM parameters given an observation sequence and a set of hidden states
- Observation likelihood can be computed using the **forward algorithm**
- Sequences of hidden states can be decoded using the **Viterbi algorithm**
- HMM parameters can be learned using the **forward-backward algorithm**